Analytical results on CDMA multiuser detection problem based on statistical mechanics

Toshiyuki Tanaka
Department of Electronics and Information Engineering
Tokyo Metropolitan University
Hachioji, Tokyo, Japan

tanaka@eei.metro-u.ac.jp
Outline

- Overview
- Analysis
- Summary of results
  - Binary uniform prior, AWGN
  - Gaussian prior, AWGN (→ Linear detectors)
  - Binary non-uniform prior, AWGN
  - Binary uniform prior, Colored Gaussian noise
  - Binary uniform prior, AWGN, imperfect power control
  - LDPC-coded CDMA
Overview: Mobile communication system
Multiple access (1)

Simultaneous communication of (uncoordinated) users to the same base station

- Frequency-division (FDMA)
- Time-division (TDMA)
- Code-division (CDMA)
Multiple access (2)

How a channel is divided … ?

Frequency-division (FDMA)

Time-division (TDMA)

Code-division (CDMA)
CDMA: Principle

Alice

Signature Seq. \( \{ s_1^\mu \} \)

Binary Data \( b_1 \)
CDMA: Principle

Alice

Signature Seq. \( \{ s^A_1 \} \)

Binary Data \( b_1 \)

Bob

Signature Seq. \( \{ s^B_2 \} \)

Binary Data \( b_2 \)
CDMA: Principle

Alice

Signature Seq. \[ \{ s^μ_1 \} \]
Binary Data \( b_1 \)

Bob

Signature Seq. \[ \{ s^μ_2 \} \]
Binary Data \( b_2 \)

Noise \( \{ n^μ \} \)

Received Signal \( \{ y^μ \} \)
Base Station

\( h_1 \Rightarrow \hat{b}_1 \)
CDMA channel \((K\ \text{users})\)

Signature Seq. 
\[\{s_1^\mu\}, \{s_2^\mu\}, \ldots, \{s_K^\mu\}\]

\[b_1 \rightarrow \times \rightarrow \text{Channel}\]
\[b_2 \rightarrow \times \rightarrow \text{Channel}\]
\[\vdots \]
\[b_K \rightarrow \times \rightarrow \text{Channel}\]

Binary Data

Noise \[\{n^\mu\}\]

Received Signal \[\{y^\mu\}\]

\[y^\mu = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_k^\mu b_k + n^\mu\]

\((\mu = 1, \ldots, N)\)
CDMA detection problem

To estimate $b_1, \ldots, b_K$ from $y_1, \ldots, y_N$
**Remark 1: Relationship with Perceptron**

**Detection Problem:** To estimate $b = (b_1, \ldots, b_K)^T$ from $(s^1, y_1), \ldots, (s^N, y_N)$, where $s^{\mu} \equiv (s_1^{\mu}, \ldots, s_K^{\mu})^T$

and

$$y_{\mu} = \frac{1}{\sqrt{N}} b \cdot s^{\mu} + n^{\mu} \quad (\mu = 1, \ldots, N)$$

⇒ Problem equivalent to **Learning of Binary-weight Linear Perceptron w/ Additive Output Noise**
Analysis: Bayesian framework

- Prior: $p(b)$
- Channel chr.: $p(y|b)$
  as defined by pdf of noise $n^\mu = y^\mu - N^{-1/2}b \cdot s^\mu$
- Posterior:

$$p(b|y) = \frac{p(y|b)p(b)}{p(y)}, \quad p(y) = \sum_{b'} p(y|b')p(b')$$
Bayes decision theory

- Loss fn.:

\[ L_k(\hat{b}) = 1 - \delta_{\hat{b} b_k} \]

\( b \): True Info. Data; \( \hat{b} \): Estimate

→ **Optimum decision rule**
(in the sense of minimizing expected loss)

\[ \hat{b}_k = \arg \max_{b_k \in \{-1, 1\}} p(b_k | y), \quad p(b_k | y) = \sum_{b \setminus b_k} p(b | y) \]

**Expected loss — Bit error rate (BER)**

\[ P_b \equiv E \left( 1 - \delta_{\hat{b}_k b_k} \right) \quad \text{(same for all users due to symmetry)} \]
**Statistical-mechanical analysis**

- **Large-system limit:**
  \[ K, N \to \infty \text{ with load } \beta \equiv K/N = O(1) \]

- **Random spreading:** \( s_k^\mu \): i.i.d.; mean=0, variance=1
Replica analysis

Objective: To evaluate Shannon entropy of $y$ per user:

$$I = - \lim_{K \to \infty} \frac{1}{K} E_S \left[ \int p_0(y) \log p(y) \, dy \right]$$

- $p_0(y)$: True distribution
- $p(y)$: Postulated distribution by receiver

Replica method:

$$I = - \lim_{K \to \infty} \left\{ \lim_{n \to 0} \frac{\partial}{\partial n} \left[ \frac{1}{K} E_S \left( \int p_0(y) [p(y)]^n \, dy \right) \right] \right\}$$
Summary of results

• Binary uniform prior, AWGN
• Gaussian prior, AWGN $\rightarrow$ Linear detectors
• Binary non-uniform prior, AWGN
• Binary uniform prior, Colored Gaussian noise
• Binary uniform prior, AWGN, imperfect power control
• LDPC-coded CDMA
Summary of results

- Binary uniform prior, AWGN (Tanaka, 2002)
- Gaussian prior, AWGN → Linear detectors
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Binary uniform prior, AWGN

Assumptions:

- Binary uniform prior

\[ p(b) = \text{const. over } \{-1, 1\}^K \]

- Additive White Gaussian Noise Channel

\[ n^\mu \sim N(0, \sigma_0^2), \quad \text{i.i.d.} \]
Binary uniform prior, AWGN

Prior: \( p(b) = \text{const. over } \{-1, 1\}^K \)

Conditional:

\[
p(y \mid b) = \prod_{\mu=1}^{N} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[ -\frac{(y^\mu - N^{-1/2} \sum_{k=1}^{K} s_k^{\mu} b_k)^2}{2\sigma_0^2} \right]
\]

\( \rightarrow \) Posterior:

\[
p(b \mid y) = Z^{-1} \exp(-\sigma_0^{-2} H(b))
\]
Remark 2: Relationship with Hopfield models

\[ p(b|y) = Z^{-1} \exp(-\sigma_0^{-2} H(b)) \]

\[ H(b) \equiv \frac{1}{2} b^T W b - h^T b \]

\[ W = (w_{ij}), \quad w_{ij} = \frac{1}{N} \sum_{\mu} s_i^{\mu} s_j^{\mu} : \text{Correlation Mtx. of Signature Seq.} \]

\[ h = (h_k), \quad h_k = \frac{1}{\sqrt{N}} \sum_{\mu} s_k^{\mu} y^{\mu} : \text{Matched-Filter Output Vector} \]

\( \Rightarrow \) Ising spin systems, Hopfield models

(Miyajima et al., 1993; Kechriotis & Manolakos, 1996)
Binary uniform prior, AWGN

MPM detection:

Saddle-point equations (under RS assumption):

\[ m = \int_{-\infty}^{\infty} \tanh(\sqrt{F} z + E) \, Dz, \quad q = \int_{-\infty}^{\infty} \tanh^2(\sqrt{F} z + E) \, Dz \]

\[ E = \frac{1}{\sigma^2 + \beta(1 - q)}, \quad F = \frac{\sigma_0^2 + \beta(1 - 2m + q)}{[\sigma^2 + \beta(1 - q)]^2} \]

\[ \rightarrow P_b = Q\left(\frac{E}{\sqrt{F}}\right) \]

\[ \left( Q(z) \equiv \int_{-\infty}^{\infty} Dz, \quad Dz \equiv \frac{e^{-z^2/2} \, dz}{\sqrt{2\pi}} \right) \]
Binary uniform prior, AWGN

- IO (Individually-Optimum) Detection
  = Bayes-optimum detection (Detection at Nishimori’s temp.)

- JO (Jointly-Optimum) Detection (Verdú, 1984)
  = “Zero-temperature” detection

\[ \hat{b}^{(JO)} = \arg \min_{s \in \{-1, 1\}^K} H(b) \]
Binary uniform prior, AWGN — BER (1)

- IO (MPM, $\sigma^2 = \sigma_0^2$)  SUMF  Single-user matched filter
- JO (MPM, $\sigma^2 = 0$)  SU  Single-user bound

![Graphs showing BER for different $\beta$ values](image)

(a) $\beta = 0.02$
(b) $\beta = 1$
Binary uniform prior, AWGN — BER (2)

<table>
<thead>
<tr>
<th>IO</th>
<th>IO (MPM, $\sigma^2 = \sigma_0^2$)</th>
<th>SUMF</th>
<th>Single-user matched filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>JO</td>
<td>JO (MPM, $\sigma^2 = 0$)</td>
<td>SU</td>
<td>Single-user bound</td>
</tr>
</tbody>
</table>

\[
P_b(E_b/N_0) \quad \beta = 1.4
\]

\[
P_b(E_b/N_0) \quad \beta = 10
\]
Binary uniform prior, AWGN — Phase Diagram

\[ \beta \]

\[ E_b/N_0 \text{ [dB]} \]

(d) spinodal (IO)  

(d) spinodal (JO)  

thermo. trans.  

(JO) (IO)

AT line (JO)  

(b)  

(c)
Summary of results

- Binary uniform prior, AWGN
- **Gaussian** prior, AWGN $\rightarrow$ Linear detectors (Tanaka, 2002)
- Binary **non-uniform** prior, AWGN
- Binary uniform prior, **Colored** Gaussian noise
- Binary uniform prior, AWGN, **imperfect** power control
- LDPC-coded CDMA
Gaussian prior, AWGN (Linear detectors)

Replica analysis (Tanaka, 2002) successfully reproduces results reported in IT literature (Tse & Hanly, 1999)

Saddle-point equations (under RS assumption):

\[
m = \frac{E}{1 + E}, \quad q = \frac{E^2 + F}{(1 + E)^2}
\]

\[
E = \frac{1}{\sigma^2 + \beta(1 - q)}, \quad F = \frac{\sigma_0^2 + \beta(1 - 2m + q)}{[\sigma^2 + \beta(1 - q)]^2}
\]

\[
\rightarrow P_b = Q\left(\frac{E}{\sqrt{F}}\right)
\]
Summary of results

- Binary uniform prior, AWGN
- Gaussian prior, AWGN → Linear detectors
- Binary non-uniform prior, AWGN
  (Caire, Müller, & Tanaka, 2003)
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Binary non-uniform prior, AWGN

Significance: Analysis of iterative joint detection/decoding scheme in coded CDMA (Caire, Müller, & Tanaka, 2003)
Binary non-uniform prior, AWGN

Priors:

- True prior: \[ p_0(b_{0k} = \pm 1) = \frac{1 \pm \mu_{0k}}{2} \]

- Postulated prior: \[ p(b_k = \pm 1) = \frac{1 \pm \mu_k}{2} \]

Assumption: \[ \{(\mu_{0k}, \mu_k) | k = 1, \ldots, K\} \]: bounded, following a limiting joint dist. as \( K \to \infty \)
Binary non-uniform prior, AWGN

Saddle-point equations (under RS assumption):

\[
\begin{align*}
    m &= E_{\mu_0,\mu} \left\{ \int_{-\infty}^{\infty} \left[ \sum_{b_0=\pm1} \frac{1 + b_0 \mu_0 k}{2} \tanh \left( \sqrt{F} z + E + b_0 \tanh^{-1} \mu \right) \right] Dz \right\} \\
    q &= E_{\mu_0,\mu} \left\{ \int_{-\infty}^{\infty} \left[ \sum_{b_0=\pm1} \frac{1 + b_0 \mu_0 k}{2} \tanh^2 \left( \sqrt{F} z + E + b_0 \tanh^{-1} \mu \right) \right] Dz \right\} \\
    E &= \frac{1}{\sigma^2 + \beta (1 - q)}, \quad F = \frac{\sigma_0^2 + \beta (1 - 2m + q)}{[\sigma^2 + \beta (1 - q)]^2} \\
    \rightarrow \quad P^{(k)}_b &= \sum_{b_0 = \pm1} \frac{1 + b_0 \mu_0 k}{2} Q \left( \frac{E + b_0 \tanh^{-1} \mu_k}{\sqrt{F}} \right)
\end{align*}
\]
Summary of results

- Binary uniform prior, AWGN
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- Binary uniform prior, Colored Gaussian noise (Tanaka, 2003)

- Binary uniform prior, AWGN, imperfect power control
- LDPC-coded CDMA
Binary uniform prior, Colored Gaussian noise

Noise not i.i.d. but correlated

Bayes-optimum (IO) detection: Saddle-point equations (under RS)

\[ m = \int_{-\infty}^{\infty} \tanh(\sqrt{E}z + E) \, Dz \]

\[ E = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\omega}{S(\omega) + \beta(1 - m)} \]

\( S(\omega) (\omega \in [0, 2\pi]) \): Power spectrum of noise

\[ \rightarrow P_b = Q(\sqrt{E}) \]
Binary uniform prior, Colored Gaussian noise

Example: \( n^\mu = c n^{\mu-1} + \sigma_0 \sqrt{1 - c^2 \nu^\mu}, \quad \nu^\mu \sim N(0, 1) \)

\[
\beta = 1.8; \quad c = 0.95, 0.9, 0.8, 0.5, 0 \text{ (from left to right)}
\]
Binary uniform prior, Colored Gaussian noise

Example: \( n^\mu = c n^{\mu-1} + \sigma_0 \sqrt{1 - c^2 \nu^\mu}, \quad \nu^\mu \sim N(0, 1) \)

\[
\begin{align*}
\beta & \quad \text{vs} \quad E_b/N_0 [\text{dB}] \\
10 & \quad 1 & \quad -4 & \quad -2 & \quad 0 & \quad 2 & \quad 4 & \quad 6 & \quad 8 & \quad 10 \\
\end{align*}
\]

\( c = 0.95, 0.9, 0.8, 0.5 \) (from left to right)
Summary of results

- Binary uniform prior, AWGN
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- Binary uniform prior, AWGN, imperfect power control (Guo & Verdú, 2002)
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Binary uniform prior, AWGN, imperfect power control

Guo & Verdú (2002): Extended replica analysis to case without power control

\[ m = E_P \left[ \int_{-\infty}^{\infty} P \tanh(\sqrt{PFz} + PE) \, Dz \right] \]
\[ q = E_P \left[ \int_{-\infty}^{\infty} P \tanh^2(\sqrt{PFz} + PE) \, Dz \right] \]
\[ E = \frac{1}{\sigma^2 + \beta(1 - q)} \]
\[ F = \frac{\sigma_0^2 + \beta(1 - 2m + q)}{[\sigma^2 + \beta(1 - q)]^2} \]

\[ \rightarrow P_b^{(k)} = Q\left( \frac{P_kE}{\sqrt{P_kF}} \right) \]
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- Binary uniform prior, AWGN
- Gaussian prior, AWGN → Linear detectors
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- LDPC-coded CDMA (Tanaka & Saad, 2003)
LDPC-coded CDMA

User 1

- Encoding
- Spreading

User 2

- Encoding
- Spreading

User K

- Encoding
- Spreading

Noise

Rcvd. signal
**LDPC-coded CDMA**

- Serial concatenation of LDPC code and CDMA channel
- Regular Gallager codes used \((C, L: \text{Left/Right degree})\)
- Performance of optimum joint detection/decoding scheme
LDPC-coded CDMA

RS saddle-point equations:

\[ m = \int \tanh \left( \sqrt{F} z + E + \sum_{c=1}^{C} \tanh^{-1} \hat{x}_c \right) Dz \prod_{c=1}^{C} \left[ \hat{\pi}(\hat{x}_c) d\hat{x}_c \right] \]

\[ q = \int \tanh^2 \left( \sqrt{F} z + E + \sum_{c=1}^{C} \tanh^{-1} \hat{x}_c \right) Dz \prod_{c=1}^{C} \left[ \hat{\pi}(\hat{x}_c) d\hat{x}_c \right] \]

\[ E = \frac{1}{\sigma^2 + \beta(1 - q)} \]

\[ F = \frac{\sigma_0^2 + \beta(1 - 2m + q)}{[\sigma^2 + \beta(1 - q)]^2} \]

\[ \pi(x) = \int \delta \left[ x - \tanh \left( \sqrt{F} z + E + \sum_{c=1}^{C-1} \tanh^{-1} \hat{x}_c \right) \right] Dz \prod_{c=1}^{C-1} \left[ \hat{\pi}(\hat{x}_c) d\hat{x}_c \right] \]

\[ \hat{\pi}(\hat{x}) = \int \delta \left( \hat{x} - \prod_{i=1}^{L-1} x_i \right) \prod_{i=1}^{L-1} \left[ \pi(x_i) dx_i \right] \]
LDPC-coded CDMA

Bit error rate:

\[
P_b = \int_{-\infty}^{0} P(u) \, du
\]

\[
P(u) = \int \delta \left[ u - \tanh \left( \sqrt{F} z + E + \sum_{c=1}^{C} \tanh^{-1} \hat{x}_c \right) \right] Dz \prod_{c=1}^{C} \hat{\pi}(\hat{x}_c) \, d\hat{x}_c
\]
**LDPC-coded CDMA**

Typical structure of $P_b$-$\sigma^2$ diagram

Spinodal; Thermo. trans

![Graph showing typical structure of $P_b$-$\sigma^2$ diagram with spinodal and thermo. trans points labeled.](image-url)
LDPC-coded CDMA

Spinodal;  Thermo. trans

\[ \beta, \sigma^2 \]

(3,6) \quad (4,8)
Summary

- Introduction of CDMA multiuser detection problem
- Relationship with perceptron learning and Hopfield model
- Statistical-mechanical analysis using Replica method
References
