Bayesian image modeling by generalized sparse Markov random fields and loopy belief propagation

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Outline

1. Introduction
2. Bayesian Image Modeling by Generalized Sparse Prior
3. Noise Reductions by Generalized Sparse Prior
4. Expansions to Segmentations
5. Concluding Remarks
Assumption 1: Original images are randomly generated by according to a prior probability.

Assumption 2: Degraded images are randomly generated from the original image by according to a conditional probability of degradation process.

Bayes Formula

\[
\Pr\{\text{Original Image} \mid \text{Degraded Image}\} \propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\} \Pr\{\text{Original Image}\}
\]
Prior in Bayesian Image Modeling

Assumption: Prior Probability is given as the following Gibbs distribution with the interaction $\alpha$ between every nearest neighbour pair of pixels:

$$
\Pr\{F = f | p, \alpha\} = P(f | p, \alpha) = \frac{1}{Z(p, \alpha)} \exp\left( -\frac{1}{2} \alpha \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)
$$

$i$-th pixel

$f_i$ State Variable of Light Intensity at $i$-pixel

$f = (f_1, f_2, \ldots, f_{12})^T$ State Vector of Original Image

$V = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ Set of All the pixels

$E = \{\{1,2\},\{2,3\},\{3,4\},\{5,6\},\{6,7\},\{7,8\},\{9,10\},\{10,11\},\{11,12\},\{1,5\},\{2,6\},\{3,7\},\{4,8\},\{5,9\},\{6,10\},\{7,11\},\{8,12\}\}$ Set of All the Nearest Neighbour Pairs of Pixels

$p=0$: $q$-state Potts model

$p=2$: $q$-Ising model

(Discrete Gaussian Graphical Model)
Prior in Bayesian Image Modeling

\[
\Pr\{F = f \mid p, \alpha\} = P(f \mid p, \alpha) = \frac{1}{Z(p, \alpha)} \exp\left( -\frac{1}{2} \alpha \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)
\]

\[\alpha^* = \arg \max_{\alpha} \ln \Pr\{F = f^* \mid p, \alpha\}\]

\[u(p, \alpha) = \sum_{\{i,j\} \in E} \sum_f |f_i - f_j|^p P(f \mid p, \alpha)\]

\[u(p, \alpha^*) = u^*\]

\[u^* = \frac{1}{|E|} \sum_{\{i,j\} \in E} |f_i^* - f_j^*|^p\]

In the region of \(0 < p < 0.3504\ldots\), the first order phase transition appears and the solution \(\alpha^*\) does not exist.
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2. Bayesian Image Modeling by Generalized Sparse Prior in Noise Reductions
3. Noise Reduction Procedure by Generalized Sparse Prior
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Prior by Conditional Maximization of Entropy in Bayesian Image Modeling

**Assumption:** Prior Probability is given by the conditional maximization of entropy under constraints.

\[
\Pr\{F = f \mid p, u\} = \operatorname{arg\,max}_{P(f)} \left\{ -\sum P(z) \ln P(z) \right\}, \quad \sum \sum_{\{i,j\} \in E} |z_i - z_j|^p P(z) = u \mid E \}
\]

\[
\Pr\{\tilde{F} = \tilde{f} \mid p, u\} = P(f \mid p, \alpha(p,u)) = \frac{1}{Z(p,\alpha(p,u))} \exp \left( -\frac{1}{2} \alpha(p,u) \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)
\]

Repeat until \( A \) converges

\[
\alpha(p,u) = A \iff A \times \sqrt{\frac{1}{u\mid E|} \sum \sum_{\{i,j\} \in E} \sum f_i - f_j|^p P(f \mid p, A)}
\]

Lagrange Multiplier

Loopy Belief Propagation

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Prior Analysis by LBP and Conditional Maximization of Entropy in Bayesian Image Modeling

Prior Probability

\[
\Pr\{F = f \mid p, u\} = P(f \mid p, \alpha(p, u)) = \frac{1}{Z(p, \alpha(p, u))} \exp\left(-\frac{1}{2} \alpha(p, u) \sum_{\{i, j\} \in E} |f_i - f_j|^p\right)
\]

Repeat until \(A\) converges

\[
M_{j \rightarrow i} \leftarrow \sum_{f_j} \left( \prod_{k \in \partial j} M_{k \rightarrow j} \right) \exp\left(-\frac{1}{2} A |f_i - f_j|^p\right) \quad (\forall \{i, j\} \in E)
\]

Compute marginals \(P_{\{i, j\}}(f_i, f_j \mid p, C)\) in LBP by messages \(M\)

\[
\alpha(p, u) = A \leftarrow A \times \sqrt{\frac{1}{|E|} \sum_{\{i, j\} \in E} \sum_f |f_i - f_j|^p P_{\{i, j\}}(f_i, f_j \mid p, A)}
\]

Prior Probability

\[
\alpha(p, u) = \left(\alpha(p, u^*), u^*\right)
\]

\[
u^* = \frac{1}{|E|} \sum_{\{i, j\} \in E} (f_{i^*} - f_{j^*})^p
\]

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Prior in Bayesian Image Modeling

Log-Likelihood for $p$ when the original image $f^*$ is given

$$\ln \Pr\{F = f^* \mid p, u^*\} = -\frac{1}{2} \alpha(p, u^*) \sum_{\{i,j\} \in E} |f_i^* - f_j^*|^p$$

Free Energy of Prior

$$- \ln Z(p, \alpha(p, u^*))$$

$p^* = \arg \max_p \ln \Pr\{F = f^* \mid p, u^*\}$

$$u^* = \frac{1}{|E|} \sum_{\{i,j\} \in E} |f_i^* - f_j^*|^p$$

$LBP$ Free Energy of Prior

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Degradation Process in Bayesian Image Modeling

Assumption: Degraded image is generated from the original image by Additive White Gaussian Noise.

\[
\Pr\{\text{Original Image} \mid \text{Degraded Image}\} 
\propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\} \times \Pr\{\text{Original Image}\}
\]

\[
\Pr\{G = g \mid F = f, \sigma\} 
\propto \prod_{i \in V} \exp\left(-\frac{1}{2\sigma^2} (g_i - f_i)^2\right)
\]

\(\sigma > 0\)

- : Pixel of Original Image \(f = (f_1, f_2, \ldots, f_{12})^T\)
- : Pixel of Degraded Image \(g = (g_1, g_2, \ldots, g_{12})^T\)

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Posterior Probability and Conditional Maximization of Entropy in Bayesian Image Modeling

Posterior Probability

\[
\Pr\{F = f \mid G = g, p, \sigma, u\} = \arg \max_{P(f)} \left\{ -\sum_{z} P(z) \ln P(z) \right\}
\]

Prior Process

\[
\alpha \Pr\{\text{Degraded Image} \mid \text{Original Image}\} \propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\} \times \Pr\{\text{Original Image}\}
\]

Degradation Process

\[
\Pr\{\text{Original Image} \mid \text{Degraded Image}\}
\]

Posterior

\[
\sum \sum_{\{i, j\} \in E} (z_i - z_j)^p \ \sigma^2 = \sum \sum_{i \in V} (z_i - g_i)^2 \ \alpha(p, u) = \sum \sum_{\{i, j\} \in E} |f_i - f_j|^p
\]

Lagrange Multipliers

\[
\Pr\{F = f \mid G = g, p, \sigma, u\} = \frac{1}{Z(g, p, B, C)} \exp \left( -\frac{1}{2B} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \sum_{\{i, j\} \in E} |f_i - f_j|^p \right)
\]

Bayes Formula

\[
\sigma^2 = B \quad \text{and} \quad \alpha(p, u) = C
\]
Posterior Probability and Conditional Maximization of Entropy in Bayesian Image Modeling

$$P(f | g, p, B, C) \propto \exp \left( -\frac{1}{2B} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} C \sum_{\{i, j\} \in E} |f_i - f_j|^p \right)$$

$$P(f | p, C) \propto \exp \left( -\frac{1}{2} C \sum_{\{i, j\} \in E} |f_i - f_j|^p \right)$$

Deterministic Equations for $B$, $C$ and $u$

$$\frac{1}{|E|} \sum_{\{i, j\} \in E} \sum_{f} |f_i - f_j|^p \quad P(f | g, p, B, C) = \frac{1}{|E|} \sum_{\{i, j\} \in E} \sum_{f} |f_i - f_j|^p \quad P(f | p, C) = u$$

$$\frac{1}{|V|} \sum_{f} (f_i - g_i)^2 \quad P(f | g, p, B, C) = B$$
Noise Reduction Procedures Based on LBP and Conditional Maximization of Entropy

Input $p$ and data $g$

Repeat until $C$ and $M$ converge

\[
P_{i,j}(f_i, f_j | p, C) \Leftarrow M_{j \rightarrow i} \Leftarrow \sum_{f_j} \left( \prod_{k \in \partial j \setminus i} M_{k \rightarrow j} \right) \exp \left( -\frac{1}{2} C |f_i - f_j|^p \right)
\]

\[
C \Leftarrow C \times \sqrt{\frac{1}{u|E|} \sum_{(i,j) \in E} \sum_{f_i} \sum_{f_j} |f_i - f_j|^p P_{i,j}(f_i, f_j | p, C)}
\]

\[
L_{j \rightarrow i} \Leftarrow \sum_{f_j} \left( \prod_{k \in \partial j \setminus i} L_{k \rightarrow j} \right) \exp \left( -\frac{1}{2} |\partial j| B (f_j - g_j)^2 - \frac{1}{2} C |f_i - f_j|^p \right) \left( \forall i \in V \right) \left( \forall j \in \partial i \right)
\]

Compute marginals $P_{ij}(f_i, f_j | g, p, B, C)$ and $P_i(f_i | g, p, B, C)$ by messages $L$

\[
u \Leftarrow \frac{1}{|E|} \sum_{(i,j) \in E} \sum_{f_i} \sum_{f_j} |f_i - f_j|^p P_{i,j}(f_i, f_j | g, p, B, C)
\]

\[
B \Leftarrow \frac{1}{|V|} \sum_{f_i} (f_i - g_i)^2 P_i(f_i | g, p, B, C)
\]

\[
\hat{\sigma} \Leftarrow \sqrt{B}, \hat{u} \Leftarrow u, \alpha(p, \hat{u}) \Leftarrow C
\]

\[
\hat{f}_i(g, p) \Leftarrow \arg \max_p P_i(f_i | g, p, B, C)
\]

\[
\ln \Pr\{G = g \mid p, \hat{\sigma}, \hat{u}\}
\]

\[
= \ln Z(g, p, \hat{\sigma}^2, \alpha(p, \hat{u}))
\]

\[
- \ln Z(p, \alpha(p, \hat{u})) - \frac{1}{2} |V| \ln(2\pi \hat{\sigma})
\]

\[
\hat{p} = \arg \max_p \Pr\{G = g \mid p, \hat{\sigma}, \hat{u}\}
\]
Noise Reductions by Generalized Sparse Priors and Loopy Belief Propagation

Original Image

Degraded Image

Restored Image

$p=0.2$

$p=0.5$

$p=1$

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Noise Reductions by Generalized Sparse Priors and Loopy Belief Propagation

Original Image

Degraded Image

Restored Image

$p=0.2$

$p=0.5$

$p=1$

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Segmentation based on Bayesian Image Modeling by Gaussian Mixture Model

\[
P(f \mid g, \mu, \sigma, \gamma) \propto \prod_{i \in V} \gamma_{f_i} \exp \left( -\frac{1}{2} \left( \frac{g_i - \mu_{f_i}}{\sigma_{f_i}} \right)^2 \right)
\]

\[
(\hat{\mu}, \hat{\sigma}, \hat{\gamma}) = \arg \max_{\mu, \sigma, \gamma} P(g \mid \mu, \sigma, \gamma)
\]

\[
P(g \mid \mu, \sigma, \gamma) = \sum_f P(f, g \mid \mu, \sigma, \gamma)
\]

\[
= \prod_{i \in V} \left( \gamma_0 \rho_0(g_i) + \gamma_1 \rho_1(g_i) + \ldots + \gamma_{q-1} \rho_{q-1}(g_i) \right)
\]

\[
\rho_l(g_i) = \frac{1}{\sqrt{2\pi} \sigma_l} \exp \left( -\frac{1}{2} \left( \frac{g_i - \mu_l}{\sigma_l} \right)^2 \right)
\]

\(g\): Original Image

\(f\): Segmented Image
Extension of Gaussian Mixture Model based on Generalized Sparse Prior in Bayesian Segmentation

\[
P(f | g, p, C, \mu, \sigma) \propto \exp \left( -\frac{1}{2} \sum_{i \in V} \left( \frac{g_i - \mu_{f_i}}{\sigma_{f_i}} \right)^2 - \frac{1}{2} C \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)
\]

\[
\mu = (\mu_0, \mu_1, \ldots, \mu_{q-1})^T
\]

\[
\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{q-1})^T
\]

\[
P(f | p, C) \propto \exp \left( -\frac{1}{2} C \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)
\]

Deterministic Equations for C and u

\[
\frac{1}{|E|} \sum_{\{i,j\} \in E} \sum_{f} |f_i - f_j|^p P(f | g, p, C, \mu, \sigma)
\]

\[
= \frac{1}{|E|} \sum_{\{i,j\} \in E} \sum_{f} |f_i - f_j|^p P(f | p, C) = u
\]

|g: Original Image| f: Segmented Image

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Segmentation by Generalized Sparse Prior and Loopy Belief Propagation

Original Image $g$

Segmented Image $q=3, p=0.2$

Segmented Image $q=4, p=0.2$

Segmented Image $q=5, p=0.2$
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Summary

Formulation of Bayesian image modeling for image processing by means of generalized sparse priors and loopy belief propagation are proposed. Our formulation is based on the conditional maximization of entropy with some constraints. In our sparse priors, although the first order phase transitions often appear, our algorithm works well also in such cases. Numerical experimental results in noise reductions and segmentations have been shown.
References


