Statistical Learning Procedure in Loopy Belief Propagation for Probabilistic Image Processing

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References

Bayesian Network and Belief Propagation

Probabilistic Information Processing → Bayes Formula → Belief Propagation → Bayesian Network → Probabilistic Model → Graphical Model

How should we treat the calculation of the summation over $256^N$ configurations?

$$\sum_{x_1=0}^{255} \sum_{x_2=0}^{255} \cdots \sum_{x_N=0}^{255} W(x_1, x_2, \cdots, x_N)$$

It is very hard to calculate exactly except some special cases.

Formulation for approximate algorithm

Accuracy of the approximate algorithm
Contents

1. Introduction
2. Bayesian Image Analysis and Gaussian Graphical Model
3. Belief Propagation
4. Concluding Remarks
Bayesian Image Analysis

Original Image

Transmission

Degraded Image

\[
\Pr\{\text{Original Image} | \text{Degraded Image}\} = \frac{\Pr\{\text{Degraded Image} | \text{Original Image}\} \Pr\{\text{Original Image}\}}{\Pr\{\text{Degraded Image}\}}
\]

A Posteriori Probability

Degradation Process

A Priori Probability

Marginal Likelihood
Bayesian Image Analysis

Degradation Process

\[ g_i = f_i + n_i \]
\[ n_i \sim N(0, \sigma^2) \]

\[
P(\tilde{g} \mid f, \sigma) = \prod_{i \in \Omega} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2} (f_i - g_i)^2\right)
\]

Original Image

Additive White Gaussian Noise

Transmission

Degraded Image
Bayesian Image Analysis

A Priori Probability

\[ f_i \in (-\infty, +\infty) \]

\[
P(\vec{f} | \alpha) = \frac{1}{Z_{PR}(\alpha)} \prod_{ij \in N} \exp\left(-\frac{1}{2} \alpha (f_i - f_j)^2\right)
\]

Generate

Standard Images

Similar?
Bayesian Image Analysis

A Posteriori Probability

\[
P(\tilde{f}|\tilde{g}, \alpha, \sigma) = \frac{P(\tilde{g}|\tilde{f}, \sigma)P(\tilde{f}|\alpha)}{P(\tilde{g}|\alpha, \sigma)}
\]

\[
= \frac{1}{Z_{\text{POS}}(\tilde{g}, \alpha, \sigma)}
\]

\[
\times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i \in \Omega} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{ij \in N} (f_i - f_j)^2 \right\}
\]

Gaussian Graphical Model
Bayesian Image Analysis

$$P(\tilde{f} | \alpha) \xrightarrow{} \tilde{f} \xrightarrow{} P(\tilde{g} | \tilde{f}, \sigma) \xrightarrow{} \tilde{g}$$

A Priori Probability
Original Image
Degraded Image

$$\tilde{f} = (f_1, f_2, \ldots, f_L)$$

$$\tilde{g} = (g_1, g_2, \ldots, g_L)$$

A Posteriori Probability

$$P(\tilde{f}, \tilde{g}, \alpha, \sigma) = \frac{P(\tilde{g} | \tilde{f}, \sigma)P(\tilde{f} | \alpha)}{P(\tilde{g} | \alpha, \sigma)}$$

$$\hat{f}_i = \int f_i P(\tilde{f} | \tilde{g}, \alpha, \sigma) df = \int_{-\infty}^{+\infty} f_i P(f_i | \tilde{g}, \alpha, \sigma) df_i$$
Hyperparameter Determination by Maximization of Marginal Likelihood

\[
(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(\tilde{g} | \alpha, \sigma)
\]

\[
\hat{f}_i = \int f_i P(\tilde{f}, \tilde{g}, \hat{\alpha}, \hat{\sigma}) df
\]

\[
P(\tilde{g} | \alpha, \sigma) = \int P(\tilde{f}, \tilde{g} | \alpha, \sigma) df = \int P(\tilde{g} | \tilde{f}, \sigma) P(\tilde{f} | \alpha) df
\]

\[
P(\tilde{f} | \alpha) \rightarrow \tilde{f} \rightarrow P(\tilde{g} | \tilde{f}, \sigma) \rightarrow \tilde{g}
\]

Original Image
\[
\tilde{f} = (f_1, f_2, \ldots, f_L)
\]

Marginal Likelihood
\[
P(\tilde{g} | \alpha, \sigma) \rightarrow \tilde{g}
\]

Degraded Image
\[
\tilde{g} = (g_1, g_2, \ldots, g_L)
\]
Maximization of Marginal Likelihood by EM (Expectation Maximization) Algorithm

Marginal Likelihood

\[ P(\tilde{g}|\alpha,\sigma) = \int P(\tilde{g}|f,\sigma)P(f|\alpha)df \]

\[ Q(\alpha',\sigma'|\alpha,\sigma,\tilde{g}) = \int P(f|\tilde{g},\alpha,\sigma)\ln P(f,\tilde{g}|\alpha',\sigma')df \]

\[ \frac{\partial}{\partial \sigma} P(\tilde{g}|\alpha,\sigma) = 0, \quad \frac{\partial}{\partial \alpha} P(\tilde{g}|\alpha,\sigma) = 0 \]

Incomplete Data

\[ \tilde{g} = (g_1, g_2, \cdots, g_L) \]

Equivalent

\[ \left[ \frac{\partial}{\partial \alpha'} Q(\alpha',\sigma'|\alpha,\sigma,\tilde{g}) \right]_{\alpha' = \alpha, \sigma' = \sigma} = 0, \quad \left[ \frac{\partial}{\partial \sigma'} Q(\alpha',\sigma'|\alpha,\sigma,\tilde{g}) \right]_{\alpha' = \alpha, \sigma' = \sigma} = 0 \]
Maximization of Marginal Likelihood by EM (Expectation Maximization) Algorithm

Marginal Likelihood

\[
P(\tilde{g} | \alpha, \sigma) = \int P(\tilde{g} | \tilde{f}, \sigma) P(\tilde{f} | \alpha) d\tilde{f}
\]

Q-Function

\[
Q(\alpha', \sigma' | \alpha, \sigma, \tilde{g}) = \int P(\tilde{f} | \tilde{g}, \alpha \alpha, \sigma) \ln P(\tilde{f}, \tilde{g} | \alpha', \sigma') d\tilde{f}
\]

EM Algorithm

Iterate the following EM-steps until convergence:

E - Step : \(Q(\alpha', \sigma' | \alpha(t), \sigma(t)) \leftarrow \int P(\tilde{f} | \tilde{g}, \alpha(t), \sigma(t)) \ln P(\tilde{f}, \tilde{g} | \alpha', \sigma') d\tilde{f}\).

M - Step : \((\alpha(t+1), \sigma(t+1)) \leftarrow \arg \max_{(\alpha', \beta')} Q(\alpha', \sigma' | \alpha(t), \sigma(t))\).

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Probabilistic Model on a Graph with Loops

\[ P(x_1, x_2, \cdots, x_L) \equiv \frac{1}{Z} \prod_{ij \in N} W_{ij}(x_i, x_j) \]

\[ Z \equiv \sum_{x_1} \sum_{x_2} \cdots \sum_{x_L} \prod_{ij \in N} W_{ij}(x_i, x_j) \]

Marginal Probability

\[ P_1(x_1) \equiv \sum_{x_2} \sum_{x_3} \sum_{x_4} \cdots \sum_{x_L} P(x_1, x_2, x_3, x_4, \cdots, x_L) \]

\[ P_{12}(x_1, x_2) \equiv \sum_{x_3} \sum_{x_4} \cdots \sum_{x_L} P(x_1, x_2, x_3, x_4, \cdots, x_L) \]
Belief Propagation

\[ P_1(\xi) = \sum_\zeta P_{12}(\xi, \zeta) \]

Message Update Rule

\[ M_{1\rightarrow 2}(\xi) \propto \sum_\zeta W_{12}(\zeta, \xi) M_{3\rightarrow 1}(\zeta) \times M_{4\rightarrow 1}(\zeta) M_{5\rightarrow 1}(\zeta) \]

\[ P_1(x_1) \equiv \frac{1}{Z_1} M_{2\rightarrow 1}(x_1) M_{3\rightarrow 1}(x_1) \times M_{4\rightarrow 1}(x_1) M_{5\rightarrow 1}(x_1) \]

\[ P_{12}(x_1, x_2) \equiv \frac{1}{Z_{12}} M_{3\rightarrow 1}(x_1) M_{4\rightarrow 1}(x_1) M_{5\rightarrow 1}(x_1) \times W_{12}(x_1, x_2) M_{6\rightarrow 2}(x_2) M_{7\rightarrow 2}(x_2) M_{8\rightarrow 2}(x_2) \]
Message Passing Rule of Belief Propagation

$$M_{1\rightarrow 2}(\xi) = \sum_{\xi} \sum_{\xi} W_{12}(\xi, \xi) M_{3\rightarrow 1}(\xi) M_{4\rightarrow 1}(\xi) M_{5\rightarrow 1}(\xi)$$

Fixed Point Equations for Massage

$$\vec{M} = \vec{\Phi}(\vec{M})$$
Fixed Point Equation and Iterative Method

Fixed Point Equation

\[ \vec{M}^* = \Phi(\vec{M}^*) \]

Iterative Method

\[ \vec{M}_1 \leftarrow \Phi(\vec{M}_0) \]
\[ \vec{M}_2 \leftarrow \Phi(\vec{M}_1) \]
\[ \vec{M}_3 \leftarrow \Phi(\vec{M}_2) \]
\[ \vdots \]

\[ y = x \]
\[ y = \Phi(x) \]

\[ 0 \quad M^* \quad M_1 \quad M_0 \quad x \]
Image Restoration by Gaussian Graphical Model

Original Image

Degraded Image

MSE: 1512

EM Algorithm with Belief Propagation

MSE: 1529

30 November, 2005

CIMCA2005, Vienna
Comparison of Belief Propagation with Exact Results in Gaussian Graphical Model

\[ (\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(\tilde{g} | \alpha, \sigma) \]

| MSE  | \(\hat{\alpha}\)  | \(\hat{\sigma}\)  | \(\ln P(g | \hat{\alpha}, \hat{\sigma})\) |
|------|----------------|--------------|-------------------------------|
| **Belief Propagation** | 327 | 0.000611 | 36.302 | -5.19201 |
| **Exact** | 315 | 0.000759 | 37.919 | -5.21444 |

\[ \text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \tilde{f}_i)^2 \]

30 November, 2005  
CIMCA2005, Vienna
Image Restoration by Gaussian Graphical Model

Original Image

Degraded Image

Belief Propagation

Exact

MSE: 1512

MSE: 325

MSE: 315

Lowpass Filter

Wiener Filter

Median Filter

MSE: 411

MSE: 545

MSE: 447

\[ \text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2 \]
Image Restoration by Gaussian Graphical Model

Original Image  Degraded Image  Belief Propagation  Exact

MSE: 1529  MSE: 260  MSE: 236

Lowpass Filter  Wiener Filter  Median Filter

MSE: 224  MSE: 372  MSE: 244

\[ \text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2 \]
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Summary

- Formulation of belief propagation
- Accuracy of belief propagation in Bayesian image analysis by means of Gaussian graphical model (Comparison between the belief propagation and exact calculation)