Physical Fluctuomatics
11th Probabilistic image processing by means of physical models

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Textbooks


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1. Probabilistic Image Processing
2. Loopy Belief Propagation
3. Statistical Learning Algorithm
4. Summary
Assumption 1: The degraded image is randomly generated from the original image by according to the degradation process.

Assumption 2: The original image is randomly generated by according to the prior probability.

Bayes Formula:

\[
\Pr\{\text{Original Image} \mid \text{Degraded Image}\} = \frac{\Pr\{\text{Degraded Image} \mid \text{Original Image}\} \Pr\{\text{Original Image}\}}{\Pr\{\text{Degraded Image}\}}
\]
Image Restoration by Probabilistic Model

The original images and degraded images are represented by $f = (f_1, f_2, \ldots, f_{|V|})$ and $g = (g_1, g_2, \ldots, g_{|V|})$, respectively.

$\vec{r}_i = (x_i, y_i)$

Position Vector of Pixel $i$

$f_i$: Light Intensity of Pixel $i$ in Original Image

$g_i$: Light Intensity of Pixel $i$ in Degraded Image
Probabilistic Modeling of Image Restoration

Random Field of Original Image $F = (F_1, F_2, \ldots, F_{|V|})$
State Vector of Original Image $f = (f_1, f_2, \ldots, f_{|V|})$

Assumption 2: The original image is generated according to a prior probability. Prior Probability consists of a product of functions defined on the neighbouring pixels.

$$
\Pr\{\text{Original Image } f\} = \Pr\{F = f\} = \prod_{ij \in E} \Phi(f_i, f_j)
$$
Probabilistic Modeling of Image Restoration

Random Field of Original Image $F = (F_1, F_2, ..., F_{|V|})$
Random Field of Degraded Image $G = (G_1, G_2, ..., G_{|V|})$

\[
\begin{align*}
\Pr\{\text{Original Image} \mid \text{Degraded Image}\} & \quad \text{Posterior} \\
\Pr\{\text{Degraded Image} \mid \text{Original Image}\} & \quad \text{Likelihood} \\
\Pr\{\text{Original Image}\} & \quad \text{Prior}
\end{align*}
\]

Assumption 1: A given degraded image is obtained from the original image by changing the state of each pixel to another state by the same probability, independently of the other pixels.

State Vector of Original Image $f = (f_1, f_2, ..., f_{|V|})$
State Vector of Degraded Image $g = (g_1, g_2, ..., g_{|V|})$

\[
\Pr\{\text{Degraded Image } g \mid \text{Original Image } f\} = \Pr\{G = g \mid F = f\} = \prod_{i \in V} \Psi(g_i, f_i)
\]

Physical Fluctuomatics (Tohoku University)
Bayesian Image Analysis

Prior Probability $\Pr\{F = f\}$ → Original Image $f$ → Degradation Process $\Pr\{G = g | F = f\}$ → Degraded Image $g$

Posterior Probability

$$
\Pr\{F = f | G = g\} = \frac{\Pr\{G = g | F = f\} \Pr\{F = f\}}{\Pr\{G = g\}}
$$

$$
\propto \left( \prod_{i \in V} \Psi(f_i, g_i) \right) \left( \prod_{\{i, j\} \in E} \Phi(f_i, f_j) \right)
$$

$V$: Set of all the pixels

$E$: Set of all the nearest neighbour pairs of pixels

Image processing is reduced to calculations of averages, variances and co-variances in the posterior probability.
Estimation of Original Image

We have some choices to estimate the restored image from posterior probability.

In each choice, the computational time is generally exponential order of the number of pixels.

\[
\begin{align*}
(1) \hat{f} &= \arg \max_z \Pr\{F = z \mid G = g\} \\
(2) \hat{f}_i &= \arg \max_{z_i} \Pr\{F_i = z_i \mid G = g\} \\
(3) \hat{f}_i &= \arg \min_{\xi_i} \left(\xi_i - \int z_i \Pr\{F = z \mid G = g\} dz\right)^2
\end{align*}
\]

Maximum A Posteriori (MAP) estimation

Maximum posterior marginal (MPM) estimation

Thresholded Posterior Mean (TPM) estimation
Prior Probability of Image Processing

\[
\Pr \{ F = f \} = \frac{1}{Z_{\text{PR}}(\alpha)} \exp \left( -\frac{1}{2} \alpha \sum_{(i,j) \in E} (f_i - f_j)^2 \right)
\]

Digital Images
\((f_i = 0, 1, \ldots, q - 1)\)

\[
Z_{\text{PR}}(\alpha) = \sum_{z_1 = 0}^{q-1} \sum_{z_1 = 0}^{q-1} \cdots \sum_{z_{|\nu|} = 0}^{q-1} \exp \left( -\frac{1}{2} \alpha \sum_{(i,j) \in E} (z_i - z_j)^2 \right)
\]

Sampling of Markov Chain Monte Carlo Method

\(E\): Set of all the nearest-neighbour pairs of pixels

\(V\): Set of all the pixels

\(\alpha = 1\)

\(\alpha = 2\)

\(\alpha = 4\)

Paramagnetic

Ferromagnetic

Fluctuations increase near \(\alpha = 1.76\ldots\)

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Bayesian Image Analysis by Gaussian Graphical Model

Prior Probability

\( \Pr\{F = \mathbf{f} \mid \alpha\} \equiv \frac{1}{Z_{PR}(\alpha)} \exp \left( -\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2 \right) \)

\( Z_{PR}(\alpha) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (z_i - z_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|} \)

Patterns are generated by MCMC.

\( \alpha = 0.0001 \)

\( \alpha = 0.0005 \)

\( \alpha = 0.0030 \)

Markov Chain Monte Carlo Method
Degradation Process for Gaussian Graphical Model

Degradation Process is assumed to be the additive white Gaussian noise.

$$\Pr\{G = g | F = f, \sigma\} = \prod_{i \in V} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(f_i - g_i)^2\right)$$

Degraded image is obtained by adding a white Gaussian noise to the original image.

$$g_i - f_i \sim N(0, \sigma^2)$$

Histogram of Gaussian Random Numbers

Physical Fluctuomatics (Tohoku University)
Image processing is reduced to calculations of averages, variances and co-variances in the posterior probability.

Prior Probability:  $\Pr\{F = f\}$

Original Image:  $f$

Degradation Process:  $\Pr\{G = g|F = f\}$

Degraded Image:  $g$

Posterior Probability:

$$\Pr\{F = f|G = g\} = \frac{\Pr\{G = g|F = f\}\Pr\{F = f\}}{\Pr\{G = g\}}$$

$$\propto \left( \prod_{i \in V} \Psi(f_i, g_i) \right) \left( \prod_{\{i, j\} \in E} \Phi(f_i, f_j) \right)$$

$V$: Set of all the pixels

$E$: Set of all the nearest neighbour pairs of pixels

Bayesian Image Analysis
Bayesian Image Analysis

\[
\Pr\{F = f\} \rightarrow f \quad \Pr\{G = g|F = f\} \rightarrow g
\]

Prior Probability \quad Original \quad Degradation Process \quad Degraded Image

\[
\Pr\{F = f|G = g\} = \frac{\Pr\{G = g|F = f\}\Pr\{F = f\}}{\Pr\{G = g\}}
\]

\[
\hat{f}_i \leftarrow \Pr\{F_i = f_i|G = g\} = \sum_{f\setminus f_i} \Pr\{F = f|G = g\}
\]

Marginal Posterior Probability of Each Pixel

Computational Complexity
Posterior Probability for Image Processing by Bayesian Analysis

\[
\Pr\{F = f | G = g, \alpha, \sigma\} = \frac{1}{Z_{\text{Posterior}}(g, \alpha, \beta)} \exp\left( -\frac{1}{2} \beta \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right)
\]

\[
Z_{\text{Posterior}}(g, \alpha, \beta) = \sum_{z_1=0}^{q-1} \sum_{z_2=0}^{q-1} \cdots \sum_{z_{|V|}=0}^{q-1} \exp\left( -\frac{1}{2} \beta \sum_{i \in V} (z_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (z_i - z_j)^2 \right)
\]

Additive White Gaussian Noise \( \beta = \frac{1}{\sigma^2} \)

\[
\Pr\{F = f | G = g\} = \frac{1}{Z_{\text{Posterior}}(g, \alpha, \beta)} \prod_{\{i, j\} \in E} \Phi_{\{i, j\}}(f_i, f_j)
\]

\[
\Phi_{\{i, j\}}(f_i, f_j) = \exp\left( -\frac{1}{4} \beta (f_i - g_i)^2 - \frac{1}{4} \beta (f_j - g_j)^2 - \frac{1}{2} \alpha (f_i - f_j)^2 \right)
\]
Bayesian Network and Posterior Probability of Image Processing

Posterior Probability

\[
\Pr\{F = f \mid G = g\} = \frac{\Pr\{G = g \mid F = f\} \Pr\{F = f\}}{\Pr\{G = g\}}
\]

Probabilistic Model expressed in terms of Graphical Representations with Cycles

\[
\Pr\{F = f \mid G = g\} = \frac{1}{Z_{\text{Posterior}}} \prod_{\{i,j\} \in E} \Phi_{\{i,j\}}(f_i, f_j)
\]

\(V\): Set of all the pixels

\(E\): Set of all the nearest neighbour pairs of pixels
Markov Random Fields

\[
\Pr\{F = f | G = g\} = \frac{1}{Z_{\text{Posterior}}} \prod_{\{i, j\} \in E} \Phi_{\{i, j\}}(f_i, f_j)
\]

\[
\Pr\{F_i = f_i | F_1 = f_1, \ldots, F_{i-1} = f_{i-1}, F_{i+1} = f_{i+1}, \ldots, F_L = f_L, G = g\} \\
= \frac{\prod_{j \in \partial i} \Phi_{\{i, j\}}(f_i, f_j)}{\sum_{z_i} \prod_{j \in \partial i} \Phi_{\{i, j\}}(z_i, f_j)} = P(x_i | \{x_j | j \in \partial i\})
\]

\(\partial i: \) Set of all the nearest neighbour pixels of the pixel \(i\)
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Loopy Belief Propagation

Probabilistic Model expressed in terms of Graphical Representations with Cycles

\[
\Pr\{F = f \mid G = g\} = \frac{1}{Z_{\text{Posterior}}} \prod_{\{i,j\} \in E} \Phi_{i,j}(f_i, f_j)
\]

\(V\): Set of all the pixels

\(E\): Set of all the nearest neighbour pairs of pixels

\[
\Pr\{F_1 = f_1 \mid G = g\} \equiv \sum_{f \neq f_1} \Pr\{F = f \mid G = g\} 
\approx \frac{M_{2 \rightarrow 1}(f_1)M_{3 \rightarrow 1}(f_1)M_{4 \rightarrow 1}(f_1)M_{5 \rightarrow 1}(f_1)}{\sum_{z_1} M_{2 \rightarrow 1}(z_1)M_{3 \rightarrow 1}(z_1)M_{4 \rightarrow 1}(z_1)M_{5 \rightarrow 1}(z_1)}
\]
Loopy Belief Propagation

\[ M_{1\rightarrow 2}(f_2) = \frac{\sum_{z_1} \Phi_{\{1,2\}}(z_1, f_2) M_{3\rightarrow 1}(z_1) M_{4\rightarrow 1}(z_1) M_{5\rightarrow 1}(z_1)}{\sum_{z_1} \sum_{z_2} \Phi_{\{1,2\}}(z_1, z_2) M_{3\rightarrow 1}(z_1) M_{4\rightarrow 1}(z_1) M_{5\rightarrow 1}(z_1)} \]

Simultaneous fixed point equations to determine the messages

\[ \tilde{M} = \tilde{\Psi}(\tilde{M}) \]
Belief Propagation Algorithm for Image Processing

Step 1: Solve the simultaneous fixed point equations for messages by using iterative method

\[
M_{i \rightarrow j}(f_j) = \frac{\sum_{z_i = 0}^{q-1} \Phi_{\{i,j\}}(z_i, f_j) \prod_{k \in \delta i} M_{k \rightarrow i}(z_i)}{\sum_{z_i = 0}^{q-1} \sum_{z_j = 0}^{q-1} \Phi_{\{i,j\}}(z_i, z_j) \prod_{k \in \delta i} M_{k \rightarrow i}(z_i)}
\]

\((\forall j \in \partial i, \forall i \in V, f_i \in \{0, 1, 2, \ldots, q - 1\})\)

Step 2: Substitute the messages to approximate expressions of marginal probabilities for each pixel and compute estimated image so as to maximize the approximate marginal probability at each pixel.

\[
\Pr\{F_i = f_i | G = g\} \leftarrow \frac{\prod_{k \in \delta i} M_{k \rightarrow i}(f_i)}{\sum_{z_i = 0}^{q-1} \prod_{k \in \delta i} M_{k \rightarrow i}(f_i)} \quad (i \in V, f_i \in \{0, 1, 2, \ldots, q - 1\})
\]

\[
\hat{f}_i = \arg \max_{z_i = 0, 1, \ldots, q-1} P_i(z_i)
\]
Belief Propagation in Probabilistic Image Processing

\[ \prod_{i \in V} i \prod_{(i,j) \in E} i-j \]

\( i \) : Random Variable \( F_i \) of \( i \)-th pixel in Original Image

\( j \) : Random Variable \( G_i \) of \( i \)-th pixel in Degraded Image

\[ i = \exp\left(-\frac{1}{2}\alpha(F_i-F_j)^2\right) \]

\[ j = \exp\left(-\frac{1}{2\sigma^2}(F_i-G_i)^2\right) \]

\( F = (F_1, F_2, ..., F_{12}) \)
Random Field of Original Image

\( G = (G_1, G_2, ..., G_{12}) \)
Random Field of Degraded Image

\[ \sum_{F_j} \sum_{F_i} \]

\[ \sum_{F_i} \sum_{F_j} \]

\( (\forall \{i,j\} \in E) \) converge

Yes

No

\[ \Pr\{F_i | G = \bar{g}\} = \frac{\sum F_i}{\sum F_j} \]

\( (\forall i \in V) \)

\[ \Pr\{F_i F_j | G = \bar{g}\} = \frac{\sum F_i F_j}{\sum F_i F_j} \]

\( (\forall \{i,j\} \in E) \)
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Statistical Estimation of Hyperparameters

Hyperparameters $\alpha, \beta$ are determined so as to maximize the marginal likelihood $\Pr\{G=g|\alpha,\beta\}$ with respect to $\alpha, \beta$.

$$(\hat{\alpha}, \hat{\beta}) = \arg \max_{(\alpha,\beta)} \Pr\{G=g|\alpha,\beta\}$$

$$\Pr\{G=g|\alpha,\beta\} = \sum_z \Pr\{F=z,G=g|\alpha,\beta\}$$

$$= \sum_z \Pr\{G=g|F=z,\beta\} \Pr\{F=z|\alpha\}$$

Original Image

Degraded Image

Marginalized with respect to $F$

Marginal Likelihood

Physical Fluctuomatics (Tohoku University)
Maximum Likelihood in Signal Processing

Original Image

Parameter

Incomplete Data

Degraded Image

Data

\[ f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{|\mathcal{V}|} \end{pmatrix} \]

\[ g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{|\mathcal{V}|} \end{pmatrix} \]

Hyperparameter

\[ \alpha \]

\[ \beta \]

\[ \left( \hat{\alpha}, \hat{\beta} \right) = \arg \max_{(\alpha, \beta)} \Pr \{ G = g | \alpha, \beta \} \]

Degraded Image = Data

Marginal Likelihood

\[ \Pr \{ G = g | \alpha, \beta \} = \sum_z \Pr \{ G = g | F = z, \beta \} \Pr \{ F = z | \alpha \} \]

Extremum Condition

\[ \frac{\partial \Pr \{ G = g | \alpha, \beta \}}{\partial \alpha} \bigg|_{\alpha = \hat{\alpha}, \beta = \hat{\beta}} = 0, \quad \frac{\partial \Pr \{ G = g | \alpha, \beta \}}{\partial \beta} \bigg|_{\alpha = \hat{\alpha}, \beta = \hat{\beta}} = 0 \]
Maximum Likelihood and EM algorithm

\[ f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{|\nu|} \end{pmatrix} \]

\[ g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{|\nu|} \end{pmatrix} \]

\[ \Pr\{G = g | \alpha, \beta\} = \sum_{z} \Pr\{G = g | F = z, \beta\} \Pr\{F = z | \alpha\} \]

Marginal Likelihood

\[ Q(\alpha, \beta | \alpha', \beta') \]

\[ Q\text{-function} \]

\[ \sum_{z} P\{F = f | G = g, \alpha', \beta'\} \ln P\{F = f, G = g | \alpha', \beta'\} \]

\[ \begin{bmatrix} \frac{\partial Q(\alpha, \beta | \alpha', \beta')}{\partial \alpha} \\ \frac{\partial Q(\alpha, \beta | \alpha', \beta')}{\partial \beta} \end{bmatrix}_{\alpha' = \alpha, \beta' = \beta} = 0 \]

\[ \begin{bmatrix} \frac{\partial \ln P\{F = f, G = g | \alpha', \beta'\}}{\partial \alpha} \\ \frac{\partial \ln P\{F = f, G = g | \alpha', \beta'\}}{\partial \beta} \end{bmatrix}_{\alpha' = \alpha, \beta' = \beta} = 0 \]

E Step: Calculate \( Q(\alpha, \beta | \alpha(t), \beta(t)) \)

M Step: Update

\[ (\alpha(t+1), \beta(t+1)) \]

\[ \left\{ \begin{array}{c} \arg \max_{(\alpha, \beta)} Q(\alpha, \beta | \alpha(t), \beta(t)) \\ \Pr\{G = g | \alpha, \beta\} \end{array} \right\}_{\alpha = \hat{\alpha}, \beta = \hat{\beta}} = 0, \quad \left\{ \begin{array}{c} \frac{\partial \Pr\{G = g | \alpha, \beta\}}{\partial \alpha} \\ \frac{\partial \Pr\{G = g | \alpha, \beta\}}{\partial \beta} \end{array} \right\}_{\alpha = \hat{\alpha}, \beta = \hat{\beta}} = 0 \]

Extremum Condition

Expectation Maximization (EM) algorithm provide us one of Extremum Points of Marginal Likelihood.
Maximum Likelihood and EM algorithm

Degrade Image $g = (g_1, g_2, ..., g_{|Y|})$

Belief Propagation

EM Algorithm

$\alpha(t+1), \alpha(t+1)$

$\arg \max_{(\sigma, \alpha)} Q(\sigma, \alpha | \sigma(t), \alpha(t), g)$

Estimates of Original Image

Gauss Markov Random Field
= Gaussian Graphical Model
$F_i \in (-\infty, +\infty)$

Binary Markov Random Field
= Ising Model
$F_i \in \{0, 1\}$

$Q(\sigma, \alpha | \sigma', \alpha', y) = \sum_f \Pr\{F = f | G = g, \sigma', \alpha'\}$

$x \log(\Pr\{F = f | G = g, \sigma, \alpha\})$

$h_\alpha(g, \hat{\sigma}, \hat{\alpha}) = \mathbb{E}[F_i | G = g, \hat{\sigma}, \hat{\alpha}]$

$h_\alpha(g, \hat{\sigma}, \hat{\alpha}) = \arg \max_{g_i \in \{0, 1\}} \Pr\{F_i = f_i | G = g, \hat{\sigma}, \hat{\alpha}\}$

Estimates of Original Images

Physical Fluctuomatics (Tohoku University)
Bayesian Image Analysis by Exact Solution of Gaussian Graphical Model

Posterior Probability

\[
Pr\{F = f \mid G = g, \alpha, \sigma\} \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in E} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2\right)
\]

Average of the posterior probability can be calculated by using the multi-dimensional Gauss integral Formula

\[
\hat{f} = \int z \Pr\{F = z \mid G = g, \alpha, \sigma\} \, dz
\]

\[
= \frac{I}{I + \alpha\sigma^2 C} \, g
\]

Multi-Dimensional Gaussian Integral Formula

\[|V|_x|V|\text{ matrix}
\]

\[
\langle i|C|j \rangle = \begin{cases} 
4 & (i = j) \\
-1 & \{i,j\} \in E \\
0 & \text{(otherwise)}
\end{cases}
\]

\[V:\text{Set of all the pixels}
\]

\[E:\text{Set of all the nearest-neighbour pairs of pixels}\]
Bayesian Image Analysis by Exact Solution of Gaussian Graphical Model

Iteration Procedure in Gaussian Graphical Model

\[ \alpha(t) \leftarrow \left( \frac{1}{|V|} \text{Tr} \left( \sigma(t-1)^2 C \right) + \frac{1}{|V|} \frac{\sigma(t-1)^2 C}{I + \alpha(t-1)\sigma(t-1)^2 C} g^T \right)^{-1} \]

\[ \sigma(t) \leftarrow \sqrt{\frac{1}{|V|} \text{Tr} \left( \sigma(t-1)^2 I \right) + \frac{1}{|V|} \frac{\alpha(t-1)^2 \sigma(t-1)^4 C^2}{I + \alpha(t-1)\sigma(t-1)^2 C} g^T} \]

\[ m(t) = \frac{I}{I + \alpha(t)\sigma(t)^2 C} g \]

\[ \hat{f}_i(t) = \arg \min_{z_i} (z_i - m_i(t))^2 \]

\[ \hat{f} \]

\[ g \]

\[ \alpha(t) \]

\[ \sigma(t) \]

Physical Fluctuomatics (Tohoku University)
Image Restorations by Exact Solution and Loopy Belief Propagation of Gaussian Graphical Model

\[
(\alpha(t + 1), \sigma(t + 1)) \leftarrow \operatorname{arg\,max}_{(\alpha, \sigma)} Q(\alpha, \sigma | \alpha(t), \sigma(t), g).
\]

\[\hat{\sigma}_{\text{Exact}} = 37.624\]

\[\hat{\sigma}_{\text{Exact}} = 0.000713\]

\[\hat{\sigma}_{\text{LBP}} = 36.335\]

\[\hat{\sigma}_{\text{LBP}} = 0.000600\]
Image Restorations by Gaussian Graphical Model

Original Image

Degraded Image

Belief Propagation

Exact

Lowpass Filter

Wiener Filter

Median Filter

MSE: 1512

MSE: 325

MSE: 315

MSE: 411

MSE: 545

MSE: 447

\[ \text{MSE} = \frac{1}{|V|} \sum |f_i - \hat{f}_i|^2 \]
**Image Restoration by Gaussian Graphical Model**

Original Image  | Degraded Image  | Belief Propagation  | Exact


- **Lowpass Filter**
  - MSE: 224

- **Wiener Filter**
  - MSE: 372

- **Median Filter**
  - MSE: 244

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Physical Fluctuomatics (Tohoku University)
Image Restoration by Discrete Gaussian Graphical Model

Original Image

\[ Q = 4 \]

Degraded Image

\[ \sigma = 1 \]

MSE: 0.98893

SNR: 1.03496 (dB)

Belief Propagation

MSE: 0.36011

SNR: 5.42228 (dB)

Physical Fluctuomatics (Tohoku University)
Image Restoration by Discrete Gaussian Graphical Model

Original Image

\[ Q = 4 \]

Degraded Image

\[ \sigma = 1 \]

MSE: 0.98893
SNR: 1.07221 (dB)

Belief Propagation

MSE: 0.28796
SNR: 6.43047 (dB)

\[ \hat{\alpha}_{LBP} = 1.02044 \]

\[ \hat{\alpha}_{LBP} = 1.20294 \]
Image Restoration by Discrete Gaussian Graphical Model

Original Image

\[ Q = 8 \]

Degraded Image

\[ \sigma = 1 \]

MSE: 0.98893
SNR: 1.89127 (dB)

Belief Propagation

MSE: 0.37697
SNR: 6.07987 (dB)

\[ \hat{\sigma}_{LBP} = 0.96380 \]
\[ \hat{\alpha}_{LBP} = 0.66033 \]
Image Restoration by Discrete Gaussian Graphical Model

Original Image

\( Q = 8 \)

Degraded Image

\( \sigma = 1 \)
MSE: 0.98893
SNR: 0.93517 (dB)

Belief Propagation
MSE: 0.32060
SNR: 5.82715 (dB)

\[ \hat{\sigma}_{LBP} = 0.85195 \]
\[ \hat{\alpha}_{LBP} = 0.62324 \]
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Summary

- Image Processing
- Bayesian Network for Image Processing
- Design of Belief Propagation Algorithm for Image Processing
Practice 10-1

For random fields \( F=(F_1,F_2,\ldots,F_{|V|})^T \) and \( G=(G_1,G_2,\ldots,G_{|V|})^T \) and state vectors \( f=(f_1,f_2,\ldots,f_{|V|})^T \) and \( g=(g_1,g_2,\ldots,g_{|V|})^T \), we consider the following posterior probability distribution:

\[
\Pr\{F = f \mid G = g\} = \frac{1}{Z_{\text{Posterior}}} \prod_{\{i,j\} \in E} \Phi_{i,j} (f_i, f_j)
\]

Here \( Z_{\text{Posterior}} \) is a normalization constant and \( E \) is the set of all the nearest-neighbour pairs of nodes. Prove that

\[
\Pr\{F_i = f_i \mid F_1 = f_1, \ldots, F_{i-1} = f_{i-1}, F_{i+1} = f_{i+1}, \ldots, F_L = f_L, G = g\} = \frac{\Pr\{F = f \mid G = g\}}{\sum_{z_i} \Pr\{F = z \mid G = g\}}
\]

and

\[
\frac{\prod_{j \in \partial i} \Phi_{i,j} (f_i, f_j)}{\sum_{z_i} \prod_{j \in \partial i} \Phi_{i,j} (z_i, f_j)}
\]

where \( \partial i \) denotes the set of all the nearest neighbour pixels of the pixel \( i \).
Make a program that generate a degraded image $g=(g_1, g_2, \ldots, g_{|V|})$ from a given image $f=(f_1, f_2, \ldots, f_{|V|})$ in which each state variable $f_i$ takes integers between 0 and $q-1$ by according to the following conditional probability distribution of the additive white Gaussian noise:

$$\Pr\{G = g | F = f\} = \prod_{i \in V} \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (f_i - g_i)^2\right)$$

Give some numerical experiments for $\sigma=20$ and 40.
Practice 10-3

Make a program for estimating $q$-valued original images $f = (f_1, f_2, \ldots, f_V)$ from a given degraded image $g = (g_1, g_2, \ldots, g_V)$ by using a belief propagation algorithm for the following posterior probability distribution for $q$ valued images:

$$\Pr\{F = f | G = g\} = \frac{1}{Z_{\text{Posterior}}} \prod_{ij \in B} \Phi_{ij}(f_i, f_j) \quad f_i \in \{0, 1, 2, \ldots, q - 1\}$$

$$\Phi_{ij}(f_i, f_j) = \exp \left( -\frac{1}{8} \beta (f_i - g_i)^2 - \frac{1}{8} \beta (f_j - g_j)^2 - \frac{1}{2} \alpha (f_i - f_j)^2 \right)$$

$$Z_{\text{Posterior}} = \sum_{z_1=0}^{q-1} \sum_{z_2=0}^{q-1} \cdots \sum_{z_V=0}^{q-1} \exp \left( -\frac{1}{8} \beta (z_i - g_i)^2 - \frac{1}{8} \beta (z_j - g_j)^2 - \frac{1}{2} \alpha (z_i - z_j)^2 \right)$$

Give some numerical experiments.


Make a program for estimating original images \( f = (f_1, f_2, \ldots, f_{|V|}) \) from a given degraded image \( g = (g_1, g_2, \ldots, g_{|V|}) \) by using a belief propagation algorithm for the following posterior probability distribution based on the Gaussian graphical model:

\[
\Pr\{F = f | G = g\} = \frac{1}{Z_{\text{Posterior}}} \prod_{\{i,j\} \in E} \Phi_{\{i,j\}}(f_i, f_j) \quad f_i \in (-\infty, +\infty)
\]

\[
\Phi_{\{i,j\}}(f_i, f_j) = \exp \left( -\frac{1}{8} \beta (f_i - g_i)^2 - \frac{1}{8} \beta (f_j - g_j)^2 - \frac{1}{2} \alpha (f_i - f_j)^2 \right)
\]

\[
Z_{\text{Posterior}} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{8} \beta (z_i - g_i)^2 - \frac{1}{8} \beta (z_j - g_j)^2 - \frac{1}{2} \alpha (z_i - z_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|}
\]

Give some numerical experiments.