Physical Fluctuomatics
5th Probabilistic information processing
by Gaussian graphical model

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Textbooks

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1. Introduction
2. Probabilistic Image Processing
3. Gaussian Graphical Model
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1. Introduction
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Markov Random Fields for Image Processing

Markov Random Fields are One of Probabilistic Methods for Image processing.


Markov Random Fields for Image Processing

In Markov Random Fields, we have to consider not only the states with high probabilities but also ones with low probabilities.

In Markov Random Fields, we have to estimate not only the image but also hyperparameters in the probabilistic model.

\[ \Rightarrow \text{We have to perform the calculations of statistical quantities repeatedly.} \]

We can calculate statistical quantities by adopting the Gaussian graphical model as a prior probabilistic model and by using Gaussian integral formulas.
Purpose of My Talk

- Review of formulation of probabilistic model for image processing by means of conventional statistical schemes.
- Review of probabilistic image processing by using Gaussian graphical model (Gaussian Markov Random Fields) as the most basic example.


Section 2 and Section 4 are summarized in the present talk.
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Bayes Formula and Bayesian Network

\[ \Pr\{A \mid B\} = \frac{\Pr\{B \mid A\} \Pr\{A\}}{\Pr\{B\}} \]

Bayes Rule

Event $B$ is given as the observed data. Event $A$ corresponds to the original information to estimate. Thus the Bayes formula can be applied to the estimation of the original information from the given data.

Prior Probability

Data-Generating Process

Bayesian Network
Image Restoration by Probabilistic Model

Assumption 1: The degraded image is randomly generated from the original image by according to the degradation process.

Assumption 2: The original image is randomly generated by according to the prior probability.

Bayes Formula

\[
\text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Marginal Likelihood}}
\]

\[
\text{Pr}\{\text{Original Image} \mid \text{Degraded Image}\} = \frac{\text{Pr}\{\text{Degraded Image} \mid \text{Original Image}\} \cdot \text{Pr}\{\text{Original Image}\}}{\text{Pr}\{\text{Degraded Image}\}}
\]
The original images and degraded images are represented by \( f = (f_1, f_2, \ldots, f_{|V|}) \) and \( g = (g_1, g_2, \ldots, g_{|V|}) \), respectively.

\[ f_i : \text{Light Intensity of Pixel } i \text{ in Original Image} \]

\[ g_i : \text{Light Intensity of Pixel } i \text{ in Degraded Image} \]

\[ \vec{r}_i = (x_i, y_i) \]

Position Vector of Pixel \( i \)

\( |V| \): Number of Pixels in Image
Probabilistic Modeling of Image Restoration

Random Field of Original Image $F = (F_1, F_2, ..., F_{|V|})$
Random Field of Degraded Image $G = (G_1, G_2, ..., G_{|V|})$

**Assumption 1:** A given degraded image is obtained from the original image by changing the state of each pixel to another state by the same probability, independently of the other pixels.

State Vector of Original Image $f = (f_1, f_2, ..., f_{|V|})$
State Vector of Degraded Image $g = (g_1, g_2, ..., g_{|V|})$

$$\Pr\{\text{Degraded Image } g \mid \text{Original Image } f \} = \Pr\{G = g \mid F = f \} = \prod_{i \in V} \Psi(g_i, f_i)$$

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Probabilistic Modeling of Image Restoration

Random Field of Original Image \( F = (F_1, F_2, \ldots, F_{|V|}) \)

State Vector of Original Image \( f = (f_1, f_2, \ldots, f_{|V|}) \)

Assumption 2: The original image is generated according to a prior probability. Prior Probability consists of a product of functions defined on the neighbouring pixels.

\[
\Pr\{\text{Original Image } f\} = \Pr\{F = f\} = \prod_{ij \in E} \Phi(f_i, f_j)
\]

Product over All the Nearest Neighbour Pairs of Pixels

Random Fields
Bayesian Image Analysis

\[ \Pr\{F = f\} \rightarrow f \rightarrow \Pr\{G = g | F = f\} \rightarrow g \]

Prior Probability \hspace{2cm} Original Image \hspace{2cm} Degradation Process \hspace{2cm} Degraded Image

Posterior Probability

\[ \Pr\{F = f | G = g\} = \frac{\Pr\{G = g | F = f\} \Pr\{F = f\}}{\Pr\{G = g\}} \]

\[ \propto \prod_{i \in V} \Psi(f_i, g_i) \prod_{\{i, j\} \in E} \Phi(f_i, f_j) \]

\[ V: \text{Set of All the pixels} \]
\[ E: \text{Set of all the nearest neighbour pairs of pixels} \]

Image processing is reduced to calculations of averages, variances and co-variances in the posterior probability.
Estimation of Original Image

We have some choices to estimate the restored image from posterior probability.

In each choice, the computational time is generally exponential order of the number of pixels.

1. \( \hat{f} = \arg \max_z \Pr\{F = z \mid G = g\} \)
   - **Maximum A Posteriori (MAP) estimation**

2. \( \hat{f}_i = \arg \max_{z_i} \Pr\{F_i = z_i \mid G = g\} \)
   - **Maximum posterior marginal (MPM) estimation**
   
   \[
   \Pr\{F_i = f_i \mid G = g\} = \sum_{f \in \mathcal{F} \setminus f_i} \Pr\{F = f \mid G = g\}
   \]

3. \( \hat{f}_i = \arg \min_{\zeta_i} \left( \zeta_i - \int z_i \Pr\{F = z \mid G = g\} dz \right)^2 \)
   - **Thresholded Posterior Mean (TPM) estimation**
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Bayesian Image Analysis by Gaussian Graphical Model

Prior Probability

\[ \Pr\{F = f \mid \alpha\} \equiv \frac{1}{Z_{PR}(\alpha)} \exp\left( -\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2 \right) \]

\[ Z_{PR}(\alpha) \equiv \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left( -\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (z_i - z_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|} \]

Patterns are generated by MCMC.

\( f_i \in (-\infty, +\infty) \)

\( \alpha = 0.0001 \quad \alpha = 0.0005 \quad \alpha = 0.0030 \)

Markov Chain Monte Carlo Method

V: Set of all the pixels

E: Set of all the nearest-neighbour pairs of pixels

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Bayesian Image Analysis by Gaussian Graphical Model

Degradation Process is assumed to be the additive white Gaussian noise.

\[ f_i, g_i \in (-\infty, +\infty) \]

\[
\Pr\{G = g | F = f, \sigma\} = \prod_{i \in V} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{1}{2\sigma^2} (f_i - g_i)^2 \right)
\]

V: Set of all the pixels

Degraded image is obtained by adding a white Gaussian noise to the original image.

\[ g_i - f_i \sim N(0, \sigma^2) \]

Histogram of Gaussian Random Numbers
Bayesian Image Analysis

Prior Probability: $\Pr\{F = f\}$

Original Image: $f$

Degradation Process: $\Pr\{G = g | F = f\}$

Degraded Image: $g$

Posterior Probability:

$$\Pr\{F = f | G = g\} = \frac{\Pr\{G = g | F = f\} \Pr\{F = f\}}{\Pr\{G = g\}}$$

$$\propto \left( \prod_{i \in V} \Psi(f_i, g_i) \right) \left( \prod_{(i, j) \in E} \Phi(f_i, f_j) \right)$$

Image processing is reduced to calculations of averages, variances and co-variances in the posterior probability.

$V$: Set of all the pixels

$E$: Set of all the nearest neighbour pairs of pixels
Bayesian Image Analysis

A Posteriori Probability

\[
P_{\text{Pr}}\{F = f \mid G = g, \alpha, \sigma\} = \frac{P_{\text{Pr}}\{G = g \mid F = f, \alpha\} P_{\text{Pr}}\{F = f \mid \alpha\}}{P_{\text{Pr}}\{G = g \mid \alpha, \sigma\}}
\]

\[
\propto \prod_{\{i,j\} \in E} \exp \left(-\frac{1}{2\sigma^2} (f_i - g_i)^2 \right) \Psi(f_i, g_i) \prod_{\{i,j\} \in E} \exp \left(-\frac{1}{2} \alpha (f_i - f_j)^2 \right) \Phi(f_i, f_j)
\]

\[
= \prod_{i \in V} \Psi(f_i, g_i) \prod_{\{i,j\} \in E} \Phi(f_i, f_j)
\]
Statistical Estimation of Hyperparameters

Hyperparameters $\alpha$, $\sigma$ are determined so as to maximize the marginal likelihood $\Pr\{G=g|\alpha,\sigma\}$ with respect to $\alpha$, $\sigma$.

$$(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} \Pr\{G = g | \alpha, \sigma\}$$

$$\hat{f} = \int z \Pr\{F = z | G = g, \hat{\alpha}, \hat{\sigma}\} \, dz$$

$$\Pr\{G = g | \alpha, \sigma\} = \int \Pr\{F = z, G = g | \alpha, \sigma\} \, dz$$

$$= \int \Pr\{G = g | F = z, \sigma\} \Pr\{F = z | \alpha\} \, dz$$

Original Image

$\Pr\{F = f | \alpha\}$

$\Pr\{G = g | F = f, \sigma\}$

Degraded Image

Marginalized with respect to $F$

$\Pr\{G = g | \alpha, \sigma\}$

Marginal Likelihood
Bayesian Image Analysis

A Posteriori Probability

\[ f_i, g_j \in (-\infty, +\infty) \]

\[
\Pr\{ F = f \mid G = g, \alpha, \sigma \} = \frac{\Pr\{ G = g \mid F = f, \alpha \} \Pr\{ F = f \mid \alpha \}}{\Pr\{ G = g \mid \alpha, \sigma \}}
\]

\[
= \frac{1}{Z_{\text{POS}}(g, \alpha, \sigma)} \exp \left( -\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right)
\]

\[
= \frac{1}{Z_{\text{POS}}(g, \alpha, \sigma)} \exp \left( -\frac{1}{2\sigma^2} (f - g)(f - g)^T - \frac{1}{2} \alpha f C f^T \right)
\]

\[
Z_{\text{POS}}(g, \alpha, \sigma) \equiv \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2\sigma^2} (z - g)(z - g)^T - \frac{1}{2} \alpha z C z^T \right) dz_1 dz_2 \cdots dz_{|V|}
\]

Gaussian Graphical Model

\[ f = (f_1, f_2, \ldots, f_{|V|}) \]
\[ g = (g_1, g_2, \ldots, g_{|V|}) \]
\[ z = (z_1, z_2, \ldots, z_{|V|}) \]

\[ |V| \times |V| \text{ matrix} \]

\[ \langle i \mid C \mid j \rangle = \begin{cases} 4 & (i = j) \\ -1 & (\{i, j\} \in E) \\ 0 & \text{(otherwise)} \end{cases} \]
Average of Posterior Probability

\[ E[F | \alpha, \sigma] = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} z \Pr\{F = z | G = g, \alpha, \sigma\} dz_1 dz_2 \cdots dz_{|V|} \]

\[ = \frac{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2\sigma^2} \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right) \left( I + \alpha \sigma^2 C \right) \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right)^T \right) dz_1 dz_2 \cdots dz_{|V|} }{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2\sigma^2} \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right) \left( I + \alpha \sigma^2 C \right) \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right)^T \right) dz_1 dz_2 \cdots dz_{|V|} } \]

\[ = \frac{I}{I + \alpha \sigma^2 C} g \]

\[ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (z - m) \exp \left( -\frac{1}{2} (z - m) (I - \alpha \sigma^2 C) (z - m)^T \right) dz_1 dz_2 \cdots dz_{|V|} = (0,0,\cdots,0) \]

Gaussian Integral formula
Bayesian Image Analysis by Gaussian Graphical Model

Posterior Probability

\[ P_r\{F = f \mid G = g, \alpha, \sigma\} \propto \exp \left( -\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right) \]

Average of the posterior probability can be calculated by using the multi-dimensional Gaussian integral Formula

\[ \hat{f} = \int z \ P_r\{F = z \mid G = g, \alpha, \sigma\} \ dz \]

\[ \frac{I}{I + \alpha \sigma^2 C} g \]

Multi-Dimensional Gaussian Integral Formula

\[ \langle i | C | j \rangle = \begin{cases} 4 & (i = j) \\ -1 & (\{i, j\} \in E) \\ 0 & \text{(otherwise)} \end{cases} \]

V: Set of all the pixels

E: Set of all the nearest-neighbour pairs of pixels

\( f_i, g_i \in (-\infty, +\infty) \)
Statistical Estimation of Hyperparameters

\[ \Pr\{G = g \mid \alpha, \sigma\} = \int \Pr\{G = g \mid F = z, \sigma\} \Pr\{F = z \mid \alpha\} dz = \frac{Z_{\text{POS}}(g, \alpha, \sigma)}{(2\pi\sigma^2)^{|V|/2} Z_{\text{PR}}(\sigma)} \]

\[ Z_{\text{POS}}(g, \alpha, \sigma) \equiv \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2} (z - g)(z - g)^T - \frac{1}{2} \alpha z C z^T\right) dz_1 dz_2 \cdots dz_{|V|} \]

\[ Z_{\text{PR}}(\alpha) \equiv \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} \alpha z C z^T\right) dz_1 dz_2 \cdots dz_{|V|} \]

Original Image

\[ \Pr\{F = f \mid \alpha\} \rightarrow f \rightarrow \Pr\{G = g \mid F = f, \sigma\} \rightarrow g \]

Marginalized with respect to \( F \)

Marginal Likelihood

\[ \Pr\{G = g \mid \alpha, \sigma\} \]
Calculations of Partition Function

\[ Z_{PR} (\alpha) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \alpha \sigma C \right) dz_1 dz_2 \cdots dz_{|V|} = \frac{(2\pi)^{|V|}}{\sqrt{\alpha^{|V|} \det C}} \]

\[ Z_{POS} (g, \alpha, \sigma) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \sigma^2 \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right) \left( I + \alpha \sigma^2 C \right) \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right)^T \right) dz_1 dz_2 \cdots dz_{|V|} \]

\[ = \exp \left( -\frac{1}{2} \sigma g \frac{C}{I + \alpha \sigma^2 C} g^T \right) \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \sigma^2 \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right) \left( I + \alpha \sigma^2 C \right) \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right)^T \right) dz_1 dz_2 \cdots dz_{|V|} \]

\[ = \sqrt{\frac{(2\pi \sigma^2)^{|V|}}{\det(I + \alpha \sigma^2 C)}} \exp \left( -\frac{1}{2} \sigma g \frac{C}{I + \alpha \sigma^2 C} g^T \right) \]

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} (z - m)^T A (z - m) \right) dz_1 dz_2 \cdots dz_{|V|} = \sqrt{\frac{(2\pi)^{|V|}}{\det A}} \]

Gaussian Integral formula  

(A is a real symmetric and positive definite matrix.)
Exact expression of Marginal Likelihood in Gaussian Graphical Model

\[
\Pr\{G = g \mid \alpha, \sigma\} = \int \Pr\{G = g \mid F = z, \sigma\} \Pr\{F = z \mid \alpha\} dz = \frac{Z_{\text{POS}}(g, \alpha, \sigma)}{(2\pi \sigma^2)^{|V|/2} Z_{\text{PR}}(\sigma)}
\]

\[
\Pr\{G = g \mid \alpha, \sigma\} = \sqrt{\frac{\det(\alpha C)}{\sqrt{(2\pi)^{|V|} \det(I + \alpha \sigma^2 C)}}} \exp\left(-\frac{1}{2} \alpha g^T \frac{C}{I + \alpha \sigma^2 C} g\right)
\]

We can construct an exact EM algorithm.

\[
(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} \Pr\{G = g \mid \alpha, \sigma\}
\]

\[
\hat{f} = \frac{I}{I + \alpha \sigma^2 C} g
\]
Bayesian Image Analysis by Gaussian Graphical Model

Iteration Procedure in Gaussian Graphical Model

\[
\alpha(t) \leftarrow \left( \frac{1}{|V|} \text{Tr} \left( \frac{\sigma(t-1)^2 C}{I + \alpha(t-1)\sigma(t-1)^2 C} \right) + \frac{1}{|V|} g \frac{C}{I + \alpha(t-1)\sigma(t-1)^2 C} g^T \right)^{-1}
\]

\[
\sigma(t) \leftarrow \sqrt{ \frac{1}{|V|} \text{Tr} \left( \frac{\sigma(t-1)^2 I}{I + \alpha(t-1)\sigma(t-1)^2 C} \right) + \frac{1}{|V|} g \frac{\alpha(t-1)^2 \sigma(t-1)^4 C^2}{(I + \alpha(t-1)\sigma(t-1)^2 C)^2} g^T }
\]

\[
m(t) = \frac{I}{I + \alpha(t)\sigma(t)^2 C} g
\]

\[
\hat{f}_i(t) = \arg \min_{z_i} (z_i - m_i(t))^2
\]
Image Restoration by Markov Random Field Model and Conventional Filters

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$$\text{MSE} = \frac{1}{|V|} \sum_{i \in V} (f_i - \hat{f}_i)^2$$

V: Set of all the pixels

Original Image          Degraded Image

Restored Image

MRF
(3x3) Lowpass
(5x5) Median

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Sample Average of Mean Square Error

\[ [\text{MSE}] \approx \frac{1}{5} \sum_{n=1}^{5} \| \vec{x} - \hat{x}_n \|^2 \]
Statistical Performance Analysis

\[ \text{MSE}(\alpha \mid \sigma, \bar{x}) = \frac{1}{|V|} \int \left\| \bar{h}(\bar{y}, \alpha, \sigma) - \bar{x} \right\|^2 \text{Pr}\{ \bar{Y} = \bar{y} \mid \bar{X} = \bar{x}, \sigma \} d\bar{y} \]

Additive White Gaussian Noise

Original Image

\( \text{Pr}\{ \bar{Y} = \bar{y} \mid \bar{X} = \bar{x}, \sigma \} \)

Restored Image

\( \text{Pr}\{ \bar{X} = \bar{x} \mid \bar{Y} = \bar{y}, \alpha, \sigma \} \)

Degraded Image

\( \text{Pr}\{ \bar{X} = \bar{x} \mid \bar{Y} = \bar{y} \} \)

Additive White Gaussian Noise

Posterior Probability

\( \bar{X} = (X_1, X_2, \ldots, X_{12})^T \)

\( \bar{Y} = (Y_1, Y_2, \ldots, Y_{12})^T \)

\( \bar{h}(\bar{y}, \alpha, \sigma) = \exp\left(-\frac{1}{2\sigma^2}(X_i - Y_i)^2\right) \)

\( \text{Pixel of Original Image} \)

\( \text{Pixel of Degraded Image} \)
Statistical Performance Analysis

\[
\text{MSE}(\alpha \mid \sigma, \bar{x}) = \frac{1}{|V|} \int \left\| \tilde{h}(\tilde{y}, \alpha, \sigma) - \bar{x} \right\|^2 \Pr\{\tilde{Y} = \tilde{y} \mid \tilde{X} = \bar{x}\} d\tilde{y}
\]

\[
= \frac{1}{|V|} \int \left\| (I + \alpha \sigma^2 C)^{-1} \tilde{y} - \bar{x} \right\|^2 \Pr\{\tilde{Y} = \tilde{y} \mid \tilde{X} = \bar{x}\} d\tilde{y}
\]

\[
\tilde{h}(\tilde{y}, \alpha, \sigma) = \int \tilde{x} P(\tilde{x} \mid \tilde{y}) d\tilde{x}
\]

\[
= (I + \alpha \sigma^2 C)^{-1} \tilde{y}
\]

\[
\langle i \mid C \mid j \rangle = \begin{cases} 
4, & i = j \in V \\
-1, & \{i, j\} \in E \\
0, & \text{otherwise}
\end{cases}
\]

\[
\Pr\{\tilde{Y} = \tilde{y} \mid \tilde{X} = \bar{x}, \sigma\} \propto \prod_{i \in V} \exp\left(-\frac{1}{2\sigma^2} (x_i - y_i)^2\right)
\]

\[
= \exp\left(-\frac{1}{2\sigma^2} \|\bar{x} - \tilde{y}\|^2\right)
\]
Statistical Performance Estimation for Gaussian Markov Random Fields

\[
\text{MSE}(\alpha \mid \sigma, \bar{x}) = \frac{1}{|V|} \left[ \left\| \frac{1}{I + \alpha \sigma^2 C} \bar{y} - \bar{x} \right\|^2 \times \Pr \{ \bar{Y} = \bar{y} \mid \bar{X} = \bar{x}, \sigma \} \right] d\bar{y}
\]

\[
= \frac{1}{|V|} \left[ \left\| \frac{1}{I + \alpha \sigma^2 C} \bar{y} - \bar{x} \right\|^2 \times \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
= \frac{1}{|V|} \left[ \left\| \frac{1}{I + \alpha \sigma^2 C} (\bar{y} - \bar{x}) + \frac{1}{I + \alpha \sigma^2 C} \bar{x} - \bar{x} \right\|^2 \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
= \frac{1}{|V|} \left[ \left\| \frac{1}{I + \alpha \sigma^2 C} (\bar{y} - \bar{x}) + \frac{\alpha \sigma^2 C}{I + \alpha \sigma^2 C} \bar{x} \right\|^2 \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
= \frac{1}{|V|} \left[ \left( \bar{y} - \bar{x} \right)^T \frac{1}{(I + \alpha \sigma^2 C)^2} (\bar{y} - \bar{x}) \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
- \frac{1}{|V|} \left[ \left( \bar{y} - \bar{x} \right)^T \frac{\alpha \sigma^2 C}{(I + \alpha \sigma^2 C)^2} \bar{x} \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
- \frac{1}{|V|} \left[ \bar{x}^T \frac{\alpha \sigma^2 C}{(I + \alpha \sigma^2 C)^2} (\bar{y} - \bar{x}) \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
+ \frac{1}{|V|} \left[ \bar{x}^T \frac{\alpha \sigma^2 C}{(I + \alpha \sigma^2 C)^2} \bar{x} \left( \frac{1}{2\pi \sigma^2} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \left\| \bar{y} - \bar{x} \right\|^2 \right) d\bar{y} \right.
\]

\[
= \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 I}{(I + \alpha \sigma^2 C)^2} \right) + \frac{1}{|V|} \bar{x}^T \frac{\alpha \sigma^2 C}{(I + \alpha \sigma^2 C)^2} \bar{x}^T
\]

\[
\langle i \mid C \mid j \rangle = \left\{ \begin{array}{ll}
4, & i = j \in V \\
-1, & \{i, j\} \in E \\
0, & \text{otherwise}
\end{array} \right.
\]

\[
= 0
\]
Statistical Performance Estimation for Gaussian Markov Random Fields

\[ \text{MSE}(\alpha \mid \sigma, \bar{x}) = \frac{1}{|V|} \int \| \nabla h(\tilde{y}, \alpha, \sigma) - \bar{x} \|^2 \times \Pr\{ \tilde{Y} = \tilde{y} \mid X = \bar{x}, \sigma \} \, d\tilde{y} \]

\[ = \frac{1}{|V|} \int \left\| \frac{I}{I + \alpha \sigma^2 \mathbf{C}} \bar{y} - \bar{x} \right\|^2 \times \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^{|V|} \exp \left( -\frac{1}{2\sigma^2} \| \bar{x} - \bar{y} \|^2 \right) d\tilde{y} \]

\[ = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 I}{(I + \alpha \sigma^2 \mathbf{C})^2} \right) + \frac{1}{|V|} \bar{x}^T \frac{\alpha^2 \sigma^4 \mathbf{C}^2}{(I + \alpha \sigma^2 \mathbf{C})^2} \bar{x} \]

\[ \langle i \mid j \rangle = \begin{cases} 4, & i = j \in V \\ -1, & \{i, j\} \in E \\ 0, & \text{otherwise} \end{cases} \]

MSE(\alpha \mid \sigma, \bar{x})

\[ \sigma=40 \]

\[ 0 \quad 0.001 \quad 0.002 \quad 0.003 \]

\[ 0 \quad 200 \quad 400 \quad 600 \]
Contents

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Summary

Formulation of probabilistic model for image processing by means of conventional statistical schemes has been summarized.

Probabilistic image processing by using Gaussian graphical model has been shown as the most basic example.


Problem 5-1: Derive the expression of the posterior probability \( \Pr\{F=f|G=g,\alpha,\sigma\} \) by using Bayes formulas \( \Pr\{F=f|G=g,\alpha,\sigma\} = \Pr\{G=g|F=f,\sigma\}\Pr\{F=f,\alpha\}/\Pr\{G=g|\alpha,\sigma\} \). Here \( \Pr\{G=g|F=f,\sigma\} \) and \( \Pr\{F=f,\alpha\} \) are assumed to be as follows:

\[
\Pr\{G = g|F = f, \sigma\} = \prod_{i \in V} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{1}{2\sigma^2} (f_i - g_i)^2 \right)
\]

\[
\Pr\{F = f | \alpha\} = \frac{1}{Z_{PR}(\alpha)} \exp\left( -\frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right)
\]

\[
Z_{PR}(\alpha) \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left( -\frac{1}{2} \alpha \sum_{\{i, j\} \in E} (z_i - z_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|}
\]

[Answer]

\[
\Pr\{F = f | G = g, \alpha, \sigma\} = \frac{1}{Z_{POS}(g, \alpha, \sigma)} \exp\left( -\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right)
\]

\[
Z_{PR}(\alpha) \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left( -\frac{1}{2\sigma^2} \sum_{i \in V} (z_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (z_i - z_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|}
\]
Problem 5-2: Show the following equality.

\[
\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 + \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2
= \frac{1}{2\sigma^2} (f - g)(f - g)^T + \frac{1}{2} \alpha \mathbf{f}^T \mathbf{C} \mathbf{f}^T
\]

\[
\langle i | C | j \rangle = \begin{cases} 
4 & (i = j) \\
-1 & (\{i, j\} \in E) \\
0 & \text{(otherwise)}
\end{cases}
\]
Problem 5-3: Show the following equality.

\[ \frac{1}{2\sigma^2} (z - g)(z - g)^T - \frac{1}{2} \alpha z C z^T \]

\[ = \frac{1}{2\sigma^2} \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right) \left( I + \alpha \sigma^2 C \right) \left( z - \frac{I}{I + \alpha \sigma^2 C} g \right)^T \]

\[ + \frac{1}{2} \alpha g \frac{C}{I + \alpha \sigma^2 C} g^T \]
Problem 5-4: Show the following equalities by using the multi-dimensional Gaussian integral formulas.

\[
Z_{\text{POS}}(g, \alpha, \sigma) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|} = \sqrt{\frac{(2\pi)^{|V|}}{|\det(I + \alpha \sigma^2 C)|}} \exp \left( -\frac{1}{2} \alpha g \begin{pmatrix} C \end{pmatrix} \begin{pmatrix} g^T \end{pmatrix} \right)
\]

\[
Z_{PR}(\alpha) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right) dz_1 dz_2 \cdots dz_{|V|} = \sqrt{\frac{(2\pi)^{|V|}}{\alpha^{|V|} |\det C|}}
\]
Problem 5-5: Derive the extremum conditions for the following marginal likelihood $\Pr\{G=g|\alpha,\sigma\}$ with respect to the hyperparameters $\alpha$ and $\sigma$.

$$\Pr\{G = g \mid \alpha, \sigma\} = \frac{\det(\alpha C)}{(2\pi)^{V/2} \det(I + \alpha \sigma^2 C)} \exp\left( -\frac{1}{2} \alpha g \frac{C}{I + \alpha \sigma^2 C} g^T \right)$$

[Answer]

$$\frac{1}{\alpha} = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 C}{I + \alpha \sigma^2 C} \right) + \frac{1}{|V|} g \frac{C}{I + \alpha \sigma^2 C} g^T$$

$$\sigma^2 = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 I}{I + \alpha \sigma^2 C} \right) + \frac{1}{|V|} g \frac{\alpha^2 \sigma^4 C^2}{(I + \alpha \sigma^2 C)^2} g^T$$
Problem 5-6: Derive the extremum conditions for the following marginal likelihood \( \Pr\{G=g|\alpha,\sigma\} \) with respect to the hyperparameters \( \alpha \) and \( \sigma \).

\[
\Pr\{G = g \mid \alpha, \sigma\} = \sqrt{\frac{\det(\alpha C)}{(2\pi)^{|V|}\det(I + \alpha\sigma^2 C)}} \exp\left(-\frac{1}{2} \alpha g \frac{C}{I + \alpha\sigma^2 C} g^T\right)
\]

[Answer]

\[
\frac{1}{\alpha} = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 C}{I + \alpha\sigma^2 C} \right) + \frac{1}{|V|} g \frac{C}{I + \alpha\sigma^2 C} g^T
\]

\[
\sigma^2 = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 I}{I + \alpha\sigma^2 C} \right) + \frac{1}{|V|} g \frac{\alpha^2 \sigma^4 C^2}{(I + \alpha\sigma^2 C)^2} g^T
\]
Problem 5-7: Make a program that generate a degraded image by the additive white Gaussian noise. Generate some degraded images from a given standard images by setting $\sigma=10,20,30,40$ numerically. Calculate the mean square error (MSE) between the original image and the degraded image.

$$\text{MSE} = \frac{1}{|V|} \sum_{i \in V} (f_i - g_i)^2$$


Sample Program:
http://www.morikita.co.jp/soft/84661/
Problem 5-8: Make a program of the following procedure in probabilistic image processing by using the Gaussian graphical model and the additive white Gaussian noise.

Algorithm:
Repeat the following procedure until convergence

\[
\alpha(t) \leftarrow \frac{1}{|V|} \text{Tr} \left( \frac{\sigma(t-1)^2 C}{I + \alpha(t-1) \sigma(t-1)^2 C} \right) + \frac{1}{|V|} g C \left( I + \alpha(t-1) \sigma(t-1)^2 C \right)^{-1} g^T
\]

\[
\sigma(t) \leftarrow \sqrt{\frac{1}{|V|} \text{Tr} \left( \frac{\sigma(t-1)^2 I}{I + \alpha(t-1) \sigma(t-1)^2 C} \right) + \frac{1}{|V|} g \frac{\alpha(t-1)^2 \sigma(t-1)^4 C^2}{(I + \alpha(t-1) \sigma(t-1)^2 C)^2} g^T}
\]

\[
m(t) \leftarrow \frac{I}{I + \alpha(t) \sigma(t)^2 C} g
\]

\[
\hat{f}_i(t) \leftarrow \arg \min_{z_i} (z_i - m_i(t))^2
\]


Sample Program: http://www.morikita.co.jp/soft/84661/