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Average Stopping Set Weight Distribution of Redundant Random Matrix Ensembles

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1 Introduction

On binary erasure channel (BEC), the decoding performance of belief propagation (BP)-based iterative decoder for low-density parity-check(LDPC) codes is dominated by combinatorial structures in a Tanner graph, which is called *stopping sets* (SS)[1]. Di et al. introduced the idea of the stopping sets and presented a recursive method to evaluate the average block and bit error probabilities of LDPC codes[7] of finite length on BEC[1]. Orlitsky et al. [2] showed asymptotic behavior of SS weight distributions and extended the results by Di et al. to the irregular code case.

For a given binary linear code C, it is hoped to find the best representation of C (i.e., a parity check matrix) which yields the smallest block (or bit) error probability when it is decoded with iterative decoding on BEC. A parity check matrix which defines C can be a *redundant parity check matrix* which is not a full-rank matrix; namely, it can contain some linearly dependent rows. For example, some finite geometry LDPC codes require a redundant parity check matrix to achieve good decoding performance with BP. Recent works of Schwartz and Vardy[3], Abdel-Ghaffar and Weber[4], Hollmann and Tolhuizen[5] show the stopping set weight distribution of a given code can be improved by using a redundant parity check matrix including linearly dependent rows.

The average stopping set weight distributions presented in [1] and [2] are useful decoding performance measure(with BP on BEC) of an instance of a given ensemble of parity check matrices. The distribution can be used for optimizing an ensemble which is suitable for BEC. BEC is not only practically interesting but also can be considered as a good starting point of theoretical studies to performance analysis on BP for more general channels.

In this paper, redundant random matrix ensembles (abbreviated as *redundant random ensembles*) are defined and its SS weight distributions are analyzed. The redundant random ensemble consists of set of binary matrices with linearly dependent rows. These linearly dependent rows (redundant rows) significantly reduces the number of stopping sets with small size. An upper bound and a lower bound on the average SS weight distribution of the redundant random ensemble will be shown.

2 Average SS weight distribution

In this section, some notations and definitions required in the paper are introduced.

2.1 Stopping set and SS weight distribution

Let F_2 be the binary Galois field with elements $\{0, 1\}$. The operator $\circ$ denotes the integer ring inner product defined by $\boldsymbol{h} \circ \boldsymbol{x} \stackrel{\triangle}{=} h_1x_1 + h_2x_2 + \cdots + h_nx_n$ for $\boldsymbol{h} \stackrel{\triangle}{=} (h_1, h_2, \ldots, h_N) \in F_2^n$ and $\boldsymbol{x} \stackrel{\triangle}{=} (x_1, x_2, \ldots, x_n) \in F_2^n$ ($n \geq 1$). The additions in the above definition of $\circ$ is the addition of the integer ring (i.e, $1 + 1 = 2$). In this paper, the addition of F_2 is denoted by $\oplus$ (i.e, $1 \oplus 1 = 0$).

For a given $\boldsymbol{x} \in F_2^n$ and an $m \times n$ binary matrix H ($m, n \geq 1$), the *SS indicator* $q_H(\boldsymbol{x})$ is defined by

$$q_H(\boldsymbol{x}) \stackrel{\triangle}{=} \#\{i \in [1, m] : \boldsymbol{h}_i \circ \boldsymbol{x} = 1\}, \tag{1}$$

where $\boldsymbol{h}_i$ denotes the i-th row vector of H. The cardinality of a given finite set X is denoted by $\#X$ in this paper. The notation $[a, b]$ means the set of consecutive integers from a to b. The stopping set is defined as follows.

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Definition 1 (Stopping set) *If $q_H(\boldsymbol{x}) = 0$ holds, then $\boldsymbol{x} \in F_2^n$ is called a* SS vector *of H. The support set of $\boldsymbol{x}$*

$$S_{\boldsymbol{x}} \stackrel{\triangle}{=} \{i \in [1,n] : x_i = 1\} \tag{2}$$

is called a stopping set of H[2]. □

Note that, if there exist a row vector $\boldsymbol{h}_i (i \in [1,m])$ satisfying $\boldsymbol{h}_i \circ \boldsymbol{x} = 1$, $\boldsymbol{x}$ is not a SS vector. Let $\boldsymbol{y} \stackrel{\triangle}{=} (y_1, y_2, \ldots, y_n) \in \{0,1,e\}^n$ be a received word through a BEC channel, where e denotes the erasure symbol. It is known that BP fails to decode $\boldsymbol{y}$ if and only if the erasure support set $E_{\boldsymbol{y}} \stackrel{\triangle}{=} \{i \in [1,n] : y_i = e\}$ contains a non-empty stopping set of H. This property justifies studies on SS which reveals the BP decoding performance for BEC.

The next definition gives the definition of the SS weight distribution and the stopping distance.

Definition 2 (SS weight distribution) *For a given $m \times n (m, n \geq 1)$ matrix H, the SS weight distribution $\{S_w(H)\}_{w=0}^n$ is defined by*

$$S_w(H) \stackrel{\triangle}{=} \sum_{\boldsymbol{x} \in Z^{(n,w)}} I[q_H(\boldsymbol{x}) = 0] \tag{3}$$

for $0 \leq w \leq n$ where $Z^{(n,w)}$ is the set of constant weight binary vectors of length n whose Hamming weight are w. The notation $I[condition]$ is the indicator function such that $I[condition] = 1$ if condition is true; otherwise, it gives 0. The stopping distance of H is defined by

$$\Delta(H) \stackrel{\triangle}{=} \min\{w \in [1,n] : S_w(H) \neq 0\}. \tag{4}$$

□

Example 1 *Let*

$$H = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{5}$$

In this case, $W \stackrel{\triangle}{=} \{\emptyset, \{1,3\}, \{1,2,3\}\}$ is the set of stopping sets of H and $2^{[1,n]} \backslash W$ is the set of non-stopping set of H. The SS weight distribution is given by $\{S_w(H)\}_{w=0}^3 = \{1,0,1,1\}$ and the stopping distance is $\Delta(H) = 2$. □

2.2 Average SS weight distribution

Suppose that $\mathcal{G}$ is a set of binary $m \times n$ matrices$(1 \leq m, n)$. Note that we allow that $\mathcal{G}$ contains some matrices with same configuration. Such matrices should be distinguished as distinct matrices. We assign equal probability $1/\mathcal{G}$ to each matrix in $\mathcal{G}$. Let $f(H)$ be a real-valued function which depends on $H \in \mathcal{G}$. The expectation of $f(H)$ with respect to the ensemble $\mathcal{G}$ is defined by

$$E_{\mathcal{G}}[f(H)] \stackrel{\triangle}{=} \sum_{H \in \mathcal{G}} P(H) f(H) = \frac{1}{\#\mathcal{G}} \sum_{H \in \mathcal{G}} f(H). \tag{6}$$

The average SS weight distribution is defined as follows:

Definition 3 *The average SS weight distribution of a given ensemble $\mathcal{G}$ is defined by*

$$S_w^{\mathcal{G}} \stackrel{\triangle}{=} E_{\mathcal{G}}[S_w(H)] \tag{7}$$

for $0 \leq w \leq n$. □

One of the most important properties of an ensemble is its symmetry. Although several types of symmetry are shown to be useful in [8], the following simple definition is enough for this paper.

[2]This definition of SS is not exactly same as the original definition. The definition covers the case where there exists a variable node without edge(i.e., zero column).

Definition 4 (Symmetry) *If the equality*

$$\#\{H \in \mathcal{G} : q_H(\boldsymbol{x}_1) = 0\} = \#\{H \in \mathcal{G} : q_H(\boldsymbol{x}_2) = 0\} \tag{8}$$

holds for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in Z^{(n,w)}$ *and any* $w \in [0, n]$, *then the ensemble* $\mathcal{G}$ *is called* symmetric. □

A symmetric ensemble has a simple expression of the average SS weight distribution. The next lemma shows that evaluation of the average SS weight distribution is equivalent to a counting problem of matrices satisfying a certain condition.

Lemma 1 *If* $\mathcal{G}$ *is symmetric, then*

$$S_w^{\mathcal{G}} = \frac{\binom{n}{w}}{\#\mathcal{G}} \#\{H \in \mathcal{G} : q_H(\boldsymbol{x}) = 0\} \tag{9}$$

holds for $w \in [0, n]$ *and any* $\boldsymbol{x} \in Z^{(n,w)}$ [3]. □

The following two ensembles are important in this paper.

Definition 5 (Random ensemble) *The random ensemble* $\mathcal{R}_{m,n}$ *contains all the binary* $m \times n$ *matrices* $(m, n \geq 1)$. *Thus, the size of* $\mathcal{R}_{m,n}$ *is equal to* 2^{mn}. □

Definition 6 (Constant row weight ensemble)
The constant row weight ensemble $\mathcal{C}_{m,n,r}$ *consists of all the binary* $m \times n$ *matrices whose rows have exactly weight* r $(m, n \geq 1, r \geq 1)$. *The size of the ensemble is, thus,*

$$\#\mathcal{C}_{m,n,r} = \binom{n}{r}^m .$$

□

3 Redundant Extension and Extended Ensembles

In this section, we consider extended ensembles obtained from the random ensemble.

3.1 Redundant extension

Before starting discussion on redundant extension, it may worth seeing how some stopping sets can be eliminated by extending a matrix.

Example 2 *Consider the following matrix:*

$$H \triangleq \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}. \tag{10}$$

It is easy to see that $\{2, 3, 4\}$ *is a stopping set (namely the sub-matrix composed from 2,3,4th columns has no row with weight 1) . Appending (0 0 0 1) (obtained by adding the first and second rows of* H*) to* H *as a row vector, we have a modified matrix* H'*:*

$$H' \triangleq \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

We can observe that the weight of the last row of the sub-matrix corresponding to 2,3,4th columns is 1. It implies that $\{2, 3, 4\}$ *is no longer a stopping set of* H'*. Note that the row spaces spanned by* H *and* H' *are exactly same.* □

[3] Due to space limitation, all the lemmas and theorems reported in this paper will be shown without proof. The proofs will be given in the forthcoming full paper version.

The previous example shows the possibility that the SS weight distribution could be improved with adding linearly dependent rows (called *redundant rows*) to a given matrix[4].

Let H be a binary $m \times n$ $(1 \le m < n)$ matrices:

$$H = \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \vdots \\ \boldsymbol{h}_m \end{pmatrix}. \tag{12}$$

Let L be a positive integer which is a divisor of m. For $1 \le i \le 2^L - 1$, $0 \le \ell \le m/L - 1$, we define $\boldsymbol{a}_i^{(\ell)}$ by

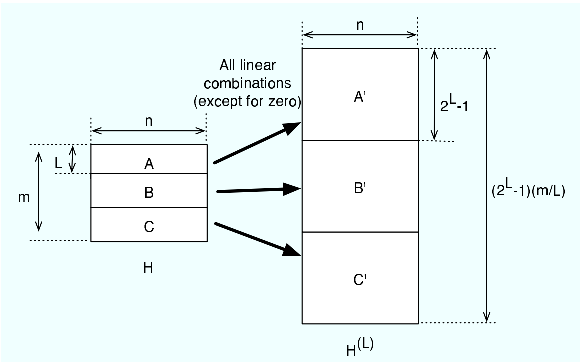
$$\boldsymbol{a}_i^{(\ell)} = (b_1(i), b_2(i), \ldots, b_L(i)) \begin{pmatrix} \boldsymbol{h}_{L\ell+1} \\ \boldsymbol{h}_{L\ell+2} \\ \vdots \\ \boldsymbol{h}_{L\ell+L} \end{pmatrix}, \tag{13}$$

where $b_j(i)$ is the j-th bit of binary representation of i, namely, $i = \sum_{j=1}^{L} 2^j b_j(i)$. In other words, $\boldsymbol{a}_i^{(\ell)}$ is a linear combination of $\boldsymbol{h}_{L\ell+k} (1 \le k \le L)$.

Definition 7 (Redundant extension) *The* redundant extension *of H, denoted by $H^{(L)}$, is the matrix whose row vectors are $\boldsymbol{a}_i^{(\ell)}$ for $1 \le i \le 2^L - 1$ and $0 \le \ell \le m/L - 1$. In other words, $H^{(L)}$ is given by*

$$H^{(L)} = \begin{pmatrix} \boldsymbol{h}'_1 \\ \boldsymbol{h}'_2 \\ \vdots \\ \boldsymbol{h}'_{(2^L-1)(m/L)} \end{pmatrix}, \tag{14}$$

where $\boldsymbol{h}'_{(2^L-1)\ell+i} = \boldsymbol{a}_i^{(\ell)}$ for $1 \le i \le 2^L - 1$ and $0 \le \ell \le m/L - 1$. The symbol L is called extension degree*.* □



A sub-block of H corresponds to a sub-block in $H^{(L)}$ (for example, sub-block A corresponds to sub-block A'). The row vectors in sub-block X' (X $\in$ {A,B,C}) can be obtained by making all linear combination (except for zero combination) of the row vectors in sub-block X.

Figure 1: Redundant extension of a parity check matrix

Figure 1 illustrates the idea of the redundant extension.

From this definition, the number of row vectors in $H^{(L)}$ is $(2^L - 1)(m/L)$ From the above definition of the redundant extension, it is clear that the row spaces of H and $H^{(L)}$ are same. In other words, the

[4] It is evident, from the definition of SS, that addition of redundant rows does not introduce new SS which are non-SS of the original matrix.

code defined by H coincides with the code defined by $H^{(L)}$. Although the codes defined by H and $H^{(L)}$ are same, H and $H^{(L)}$ may have different SS weight distributions.

The definition of redundant extension of a matrix naturally leads to the following definition of redundant extension of a given ensemble,

Definition 8 (Extended ensemble) *Consider the case where an ensemble $\mathcal{G}$ which consists of $m \times n$ binary matrices is given. Let L be a divisor of m. The extended ensemble of $\mathcal{G}$, denoted by $\mathcal{G}^{(L)}$, is defined by*

$$\mathcal{G}^{(L)} \triangleq \{H^{(L)} : H \in \mathcal{G}\}. \tag{15}$$

The size of the ensemble $\#\mathcal{G}^{(L)}$ equals to the size of the original ensemble $\#\mathcal{G}$. For each matrix in $\mathcal{G}^{(L)}$, equal probability is assigned. □

The *redundant random ensemble* which is the main subject of this paper is the extended ensemble of a random ensemble, which is denoted by $\mathcal{R}_{m,n}^{(L)}$.

4 Main Results

4.1 Redundant random ensemble: $L = 2$

In this subsection, we discuss the average SS weight distribution of the redundant random ensemble $\mathcal{R}_{m,n}^{(2)}$. In this case, we can derive a simple exact formula of the average SS weight distribution.

Suppose that the first 3-rows of $H \in \mathcal{R}_{m,n}^{(2)}$:

$$\tilde{H} \triangleq \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_1 \oplus \boldsymbol{h}_2 \end{pmatrix} \tag{16}$$

is given. Our first task is to count the number of pairs $(\boldsymbol{h}_1, \boldsymbol{h}_2) \in F_2^n \times F_2^n$ satisfying $q_{\tilde{H}}(\boldsymbol{x}_w) = 0$. Let us define U by

$$U \triangleq \#\{(\boldsymbol{h}_1, \boldsymbol{h}_2) \in F_2^n \times F_2^n : \boldsymbol{h}_1 \circ \boldsymbol{x}_w \neq 1, \boldsymbol{h}_2 \circ \boldsymbol{x}_w \neq 1, \\ (\boldsymbol{h}_1 \oplus \boldsymbol{h}_2) \circ \boldsymbol{x} \neq 1\}.$$

Note that $m/2$ sub-blocks exist in $H \in \mathcal{R}_{n,m}^{(2)}$ and these sub-blocks can be chosen independently when we count $\#\{H \in \mathcal{R}_{n,m}^{(2)} : q_H(\boldsymbol{x}_w) = 0\}$. This observation leads to the following equality:

$$\#\{H \in \mathcal{R}_{n,m}^{(2)} : q_H(\boldsymbol{x}_w) = 0\} = U^{m/2}. \tag{17}$$

The next lemma gives a simple description of U.

Lemma 2 *For $1 \leq m < n$, $w \geq 1$, U is given by*

$$U = (2^n - w2^{n-w})^2 - 2^{2(n-w)+1} \sum_{\gamma=2}^{w} \binom{w}{\gamma}(w - \gamma).$$

□

The following theorem is an immediate consequence of Lemma 2.

Theorem 1 *The average SS weight distribution of $\mathcal{R}_{m,n}^{(2)}$ is given by*

$$S_w^{\mathcal{R}_{m,n}^{(2)}} = \frac{\binom{n}{w}}{2^{mn}} \left((2^n - w2^{n-w})^2 - V\right)^{m/2} \tag{18}$$

for $1 \leq m < n$, $w \geq 1$, where V is defined by

$$V \triangleq 2^{2(n-w)+1} \sum_{\gamma=2}^{w} \binom{w}{\gamma}(w - \gamma). \tag{19}$$

□

4.2 Redundant random ensemble: $L > 2$

When $L > 2$, it becomes intricate to evaluate the number of extended parity check matrices which gives a stopping set for a given weight. Instead to derive an exact expression, we here utilize upper and lower bounds on the number of such parity check matrices to study the average SS weight distribution of extended ensembles.

The next theorem gives upper and lower bounds on the average SS weight distribution for $L > 2$.

Theorem 2 (Upper and lower bounds on $S_w^{\mathcal{R}_{m,n}^{(L)}}$) *The following inequalities for the average SS weight distribution of the redundant random ensemble hold:*

$$S_w^{\mathcal{R}_{m,n}^{(L)}} \geq \binom{n}{w} \max\{A^{m/L}, 2^{-m}\} \tag{20}$$

$$S_w^{\mathcal{R}_{m,n}^{(L)}} \leq \binom{n}{w} \left(\frac{1 - w2^{-w}}{(2^L-1)w2^{-w} + 1 - w2^{-w}} \right)^{m/L} \tag{21}$$

for $1 \leq w \leq n$, where A in the lower bound is defined by

$$A \stackrel{\triangle}{=} \max\{1 - (2^L - 1)2^{-w} w, 0\}. \tag{22}$$

□

It is easy to check that the upper bound and lower bound coincide with the average SS weight distribution of the non-extended ensemble.

There are trade off relations between the extension degree L and the average SS weight distribution. If L becomes large, then decoding complexity of BP-based iterative decoding gets large as well because the number of rows in the extended matrix $(2^L - 1)(m/L)$ is an exponentially increasing function of L. On the other hand, large L tends to give larger stopping distance.

4.3 Typical stopping distance

From the average SS weight distribution, we can retrieve some information on the stopping distance of matrices contained in an ensemble.

Definition 9 (Typical stopping distance) *The typical stopping distance of an ensemble $\mathcal{G}$ is defined by*

$$\delta^{\mathcal{G}} \stackrel{\triangle}{=} \min \left\{ s \in [1,n] : \sum_{w=1}^{s-1} S_w^{\mathcal{G}} \geq 1 \right\}. \tag{23}$$

□

The condition $\sum_{w=1}^{s-1} S_w(H) = 0$ is equivalent to $\Delta(H) \geq s$. It is evident that there exist a matrix $H \in \mathcal{G}$ satisfying $\sum_{w=1}^{\delta^{\mathcal{G}}-1} S_w(H) = 0$ because the average $\sum_{w=1}^{\delta^{\mathcal{G}}-1} S_w^{\mathcal{G}}$ is strictly smaller than 1. This means that there exits a matrix with stopping distance larger than or equal to the typical stopping distance $\delta^{\mathcal{G}}$.

We here compare a high rate redundant random ensemble with constant row weight ensembles in terms of typical stopping distance.

Example 3 *Consider the case where $n = 1024, m = 32$. We can show that the maximum value of the typical stopping distance of $\mathcal{C}_{n,m,r}$ is*

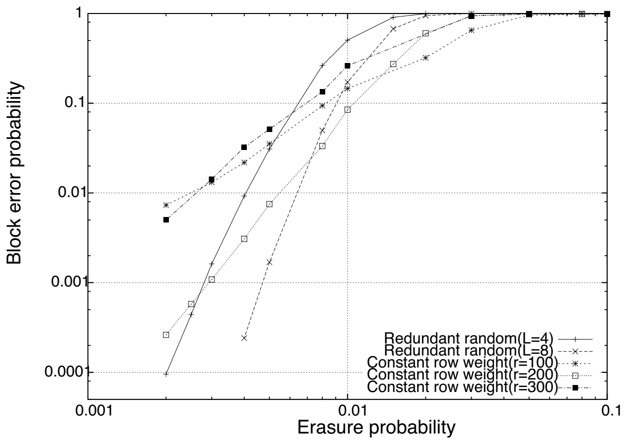
$$\max_{r \in [1,1024]} \delta^{\mathcal{C}_{1024,32,r}} = 3.$$

This result means that there are no constant row weight ensembles with $n = 1024, m = 32$ which achieves the typical stopping distance 4. On the other hand, the redundant random ensemble ($n = 1024, m = 32, L = 8$) has larger typical stopping distance:

$$\delta^{\mathcal{R}_{32,1024}^{(8)}} = 4.$$

In this case, the redundant random ensemble is expected to give asymptotically (i.e., regime of small erasure probability) better decoding performance (with BP) than the constant row weight ensemble with any row weight.

Figure 2 presents the block error probabilities of instances of the redundant random ensembles ($L = 4, 8$*) and the constant row weight ensembles (*$r = 100, 200, 300$*). Note that the size of a parity check matrix used in the BP decoder is* 120×1024 *(redundant random,* $L = 4$*),* 1020×1024 *(redundant random,* $L = 8$*),* 32×1024 *(constant row weight), respectively. We can observe that the instances of the redundant random ensembles give steeper error curves than those of the instances of the constant row weight ensembles. This difference in decoding performance could be explained from the viewpoint of typical stopping distance discussed above.* □



$n = 1000$, $m = 32$

Figure 2: Block error probabilities (BP on BEC) of instances of redundant random ensembles and constant row weight ensembles

5 Asymptotic growth rate of the average SS weight distributions of redundant random ensembles

In this section, we will discuss the asymptotic (i.e., the case where n goes to infinity) behavior of the average SS weight distribution.

5.1 Bounds on asymptotic growth rate

We will consider the asymptotic behavior of the average SS weight distribution of the redundant random ensembles.

The asymptotic growth rate $\sigma_\ell(R, \mu)$ is defined by

$$\sigma_\ell(R, \mu) \stackrel{\triangle}{=} \lim_{n \to \infty} \frac{1}{n} \log S_{\ell n}^{\mathcal{R}^{(\mu n)}_{(1-R)n,n}}, \tag{24}$$

for $0 \le \ell \le 1, 0 < \mu \le 1 - R$. The parameter μ is called *normalized extension degree*. It is evident that, from the definition of redundant extension, the above definition of $\sigma_\ell(R, \mu)$ is well defined only if $(1 - R)/\mu$ is an integer.

The next corollary gives a lower bound on $\sigma_\ell(R, \mu)$.

Corollary 1 *The asymptotic growth rate* $\sigma_\ell(R, \mu)$ *can be lower bounded by*

$$\sigma_\ell(R, \mu) \ge \begin{cases} H(\ell) - (1 - R), & \ell \le \mu \\ H(\ell), & \ell > \mu. \end{cases} \tag{25}$$

where $H(\ell)$ *is the binary entropy function.* □

The next corollary is an upper bound on $\sigma_\ell(R,\mu)$.

Corollary 2 *The asymptotic growth rate $\sigma_\ell(R,\mu)$ can be upper bounded by*

$$\sigma_\ell(R,\mu) \le \begin{cases} H(\ell) - (1-R)\left(1 - \frac{\ell}{\mu}\right), & \ell \le \mu \\ H(\ell), & \ell > \mu. \end{cases} \tag{26}$$

□

Combining the above two corollaries, we can see that $\sigma_\ell(R,\mu) = H(\ell)$ holds for $\ell > \mu$. Namely, the upper and lower bounds are asymptotically tight when $\ell > \mu$.

Example 4 *Figure 3 shows the lower bound (Corollary 1) and the upper bound (Corollary 2) for the case $R = 0.5$, $\mu = 0.25$. The horizontal axis of Fig.3 represents the normalized weight ℓ. The curve $H(\ell)$ (the asymptotic growth rate of non-extended ensemble $\mathcal{R}_{(1-R)n,n}$) is also included in Fig.3 as a reference.*

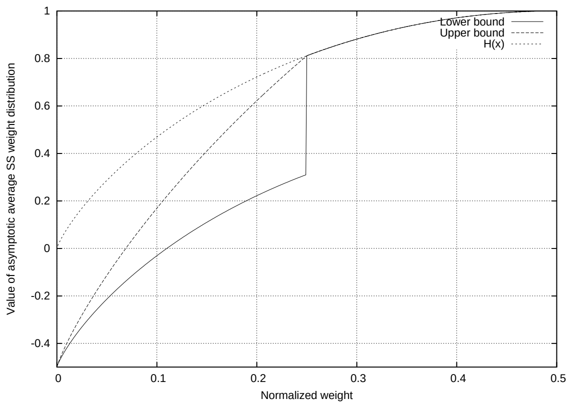


Figure 3: Upper and lower bounds on the asymptotic growth rate of the redundant random ensemble ($R = 0.5$, $\mu = 0.25$)

References

[1] C.Di, D.Proietti, I.E.Teletar, T.Richardson, R.Urbanke, "Finite-length analysis of low-density parity-check codes on the binary erasure channel," *IEEE Trans. Inform. Theory,* vol.48, pp.1570–1579, June (2002).

[2] A. Orlitsky, K. Viswanathan, and Junan Zhang, "Stopping set distribution of LDPC code ensembles," *IEEE Trans. Inform.Theory*, vol.51, no.3, pp.929–953 (2005).

[3] M. Schwartz and A. Vardy, "On the stopping distance and the stopping redundancy of codes," *IEEE Trans. Inform.Theory*, vol.52, no.3, pp.922–932 (2006).

[4] K. A. S. Abdel-Ghaffar and J.H. Weber, "On parity-check matrices with optimal stopping and/or dead-end set enumerators," in Proceedings of Turbo-coding 2006, Munich (2006).

[5] H.D.L. Hollmann and L.M.G.M. Tolhuizen, "On parity check collections for iterative erasure decoding that correct all correctable erasure patterns of a given size," arXiv: cs.IT/0507068 [Online] (2005).

[6] N. Alon and J.H. Spencer, "The probabilistic method," Wiley Inter-Science, (2000).

[7] R.G.Gallager, *"Low Density Parity Check Codes"*. Cambridge, MA:MIT Press 1963.

[8] T.Wadayama, "Average coset weight distributions of combined LDPC matrix ensembles," *IEEE Trans. Inform. Theory*, pp.4856- 4866,vol.52,no.11, Nov., 2006.