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Semi-Balanced Colorings of Graphs

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Randomness often implies uniformity, but usually there exists a much more uniform distribution than a random distribution. An interesting question is how we can construct a strongly uniform distribution for a given structure. If we have n points on a line, it is easy to color points into red and blue such that the number of red points and blue points are almost same in each interval: Indeed, it suffices to color the points into red and blue to form an alternating sequence of two colors. The theory of low-discrepancy sequence is a problem to construct an alternating sequence on-line by changing color of points from red to bule one by one such that each intermediate sequence is almost "uniform".

It is an interesting question how we can generalize this fact to a more general structure, namely, a graph. Given a graph $G = (V, E)$, a *2-coloring of* G is a mapping π from V to $\{\text{red, blue}\}$ such that $\pi(x) \neq \pi(y)$ for every edge $\{x, y\}$ in E. This is a natural extension of the alternating sequence. A graph has a 2-coloring if and only if it is bipartite; in fact, by symmetry, a bipartite graph always has two different 2-colorings.

It is a little pity that only bipartite graphs can have 2-colorings, and we want to relax the condition such that the uniformity condition holds for a wider class of graphs. A natural way to extend the notion of 2-coloring is by allowing to use k colors, where k is any fixed positive integer. Such a coloring is called a *k-coloring of* G. It is difficult question to answer what graph has a 3-coloring, for example. Indeed, it is the famous 4-coloring theorem to assert that every planar graph has a 4-coloring. The k-coloring is one of main topics in combinatorics; in particular, the number of possible k-colorings of a graph is given by its *chromatic polynomial*, and has been studied extensively (see [5]).

Another way to generalize 2-colorings is by relaxing the restriction on two adjacent vertices never being allowed to have the same color. One possible formulation is that we only require that blue nodes are not adjacent to each other: this is the famous *independent set* problem. However, we want to give a more precise way to relax the coloring condition. For this purpose, we view the 2-coloring condition as a discrepancy condition on a special hypergraph induced by the graph called the *shortest-paths hypergraph*. Discrepancy is a popular measure of uniformity

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and the quality of approximations, and has been used in combinatorics, geometry, and Monte-Carlo simulations [1, 3, 4]. It is defined as follows. Let $H = (V, \mathcal{F})$ be a hypergraph, where $\mathcal{F} \subseteq 2^V$. A mapping $\pi : V \to \{-1, 1\}$ is called a coloring of H. Given a coloring π of H, let $\pi(F) = \sum_{v \in F} \pi(v)$ for every $F \in \mathcal{F}$ and let $D_c(H, \pi) = \max_{F \in \mathcal{F}} |\pi(F)|$. The *combinatorial* (or *homogeneous*) *discrepancy* $D_c(H)$ is defined as $D_c(H) = \min_\pi D_c(H, \pi)$, where we take the minimum over all possible colorings of H. In particular, if $D_c(H, \pi) \leq 1$ then π yields a coloring which is uniform in every hyperedge; this means that $-1 \leq \pi(F) \leq 1$ for every $F \in \mathcal{F}$. Such a π is called a *balanced coloring* of H. A hypergraph does not always have a balanced coloring; indeed, H is unimodular if and only if every induced subhypergraph of H has a balanced coloring [1].

We call a coloring π *semi-balanced* if $-1 \leq \pi(F) \leq 2$ for every $F \in \mathcal{F}$. The shortest-paths hypergraph induced by G is the hypergraph $\mathcal{H}(G) = (V, \mathcal{P}_G)$, where $\mathcal{P}_G$ is the set of all shortest-path vertex sets in G. A 2-coloring of G is equivalent to a balanced coloring of $\mathcal{H}(G)$; hence, we generalize 2-colorings of G by considering semi-balanced colorings of $\mathcal{H}(G)$. We often call it a semi-balanced coloring of G. We can further define the k-balanced coloring by relaxing the condition to $-1 \leq \pi(F) \leq k$ to have a stratification of independent sets.

We show that in contrast to the fact that the number of 3-colorings of a connected graph with n vertices may be exponential in n, the number $\nu(G)$ of semi-balanced colorings is always polynomial in n. More precisely, we prove that if G is a connected graph with n vertices and m edges, the number of different semi-balanced colorings is: (1) at most $n + 1$ if G is bipartite; (2) at most m if G is non-bipartite and triangle-free; and (3) at most $m + 1$ if G is non-bipartite. Moreover, we give an $O(nm^2)$-time algorithm for enumerating all the semi-balanced colorings. This talk is based on the paper [2].

References

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