Multicritical point of the two-dimensional spin glass
– Quantum memory and analytical results –

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Some aspects of spin glasses and information processing are known to have close formal similarities [1]. In particular the relationship between the *mean-field theory* of spin glasses and processing of *classical* information is used to achieve lots of non-trivial and exciting results as exemplified by many presentations in this symposium. In my talk I will focus on mathematical analyses of the strongly *non-mean-field* two-dimensional spin glass. Some symmetry arguments including duality are fully exploited to give a conjecture on the exact location of the multicritical point in the phase diagram.

The motivation is two-fold. First, the problem of finite-dimensional spin glass is very important in its own right. Real spin glasses all live in our finite-dimensional world, and the mean-field theory (theory in the limit of infinite dimensions) is at best a *qualitatively* good description of the reality (or worse, a qualitatively incorrect picture). Second, it was pointed out recently that stable storage of *quantum* information can be achieved effectively by a new method of quantum error correction which can be mapped to the two-dimensional Ising spin glass [2]. The problem of exact location of the multicritical point is mapped to the identification of critical threshold of error correction in quantum information storage on the torus code.

Let us first write our conjecture on the exact location of the multicritical point in the phase diagram of the $\pm J$ Ising model on the two-dimensional square lattice:

$$-p_c \log p_c - (1-p_c)\log(1-p_c) = \frac{\log 2}{2}. \tag{1}$$

This formula, we believe, gives the exact critical threshold of the ferromagnetic phase on the Nishimori line (NL) in the phase diagram (figure 1) which we shall call the multicritical point.

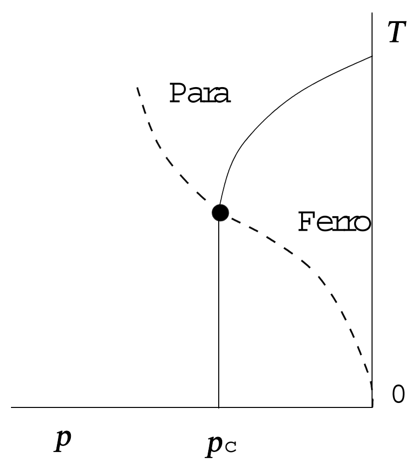

Figure 1: Phase diagram of the $\pm J$ Ising model on the square lattice. The multicritical point is shown in a black dot.

We use the replica method to treat the randomness of the $\pm J$ Ising model. Then the resulting effective partition function Z_n is shown to have many remarkable symmetry properties such as its invariance under exchanges of the edge Boltzmann factors and duality transformation. In particular, a subspace $\mathcal{N}$ generalizing the NL is shown to be globally self-dual when the number of replicas n

is even. This gives us a good (mathematical in addition to physical) reason to focus our attention to the critical properties on the NL, the multicritical point, instead of investigating the whole phase diagram. Thus we discuss invariant (fixed-point) conditions of the system on the NL under duality transformation.

We find a set of candidates, fixed-point conditions, that may possibly be identified with the multicritical point. A rigorous inequality derived from the dual representation of the replicated system and numerical simulations are used to exclude most of these candidates, resulting in the following representation of the possible multicritical point for n an arbitrary integer:

$$e^{(n+1)K} + e^{-(n+1)K} = 2^{-n/2}(e^K + e^{-K})^{n+1}. \tag{2}$$

The quenched limit ($n \to 0$) in conjunction with the NL condition leads to (1).

To check if (2) is correct for integer n, we first confirmed that this is indeed exact for $n = 1$, 2 and ∞. Next we have performed an extensive Monte Carlo simulation for $n = 3$ using the non-equilibrium relaxation method and have found that $T_c = 1.660(5)$ for the multicritical point (figure 2); formula (2) gives 1.65858, in good agreement. Several other possibilities are ruled out from these data. Also,

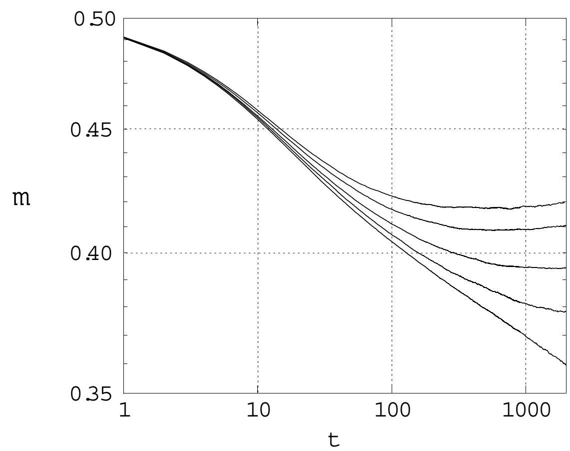

Figure 2: Relaxation of magnetization with mixed phase initialization when $n = 3$. The temperatures are 1.650, 1.655, 1.660, 1.663, 1.665 from top to bottom.

numerical investigations are abundant for the quenched limit ($n \to 0$), and all the available numbers support our conjecture.

We therefore believe that most of the candidates of the exact location of the multicritical point have been checked and excluded excepting (2) although there still remains the possibility that we have not exhausted all of them.

A preliminary report appeared in [3] and a full account is in preparation [4]. The present work was done in collaboration with Koji Nemoto and Jean-Marie Maillard.

References

[1] H. Nishimori: *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford Univ. Press, Oxford, 2001).

[2] E. Dennis, A. Kitaev, A. Landahl and J. Preskill, J. Math. Phys. **43** (2002) 4452; C. Wang, J. Harrington and J. Preskill, Ann. Phys. **303** (2003) 31.

[3] H. Nishimori and K. Nemoto, J. Phys. Soc. Jpn. **71** (2002) 1198.

[4] J.-M. Maillard, K. Nemoto and H. Nishimori, (in preparation).