Probabilistic image processing and Bayesian network

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References
Bayesian Network and Belief Propagation

Probabilistic Information Processing

Bayes Formula

Belief Propagation

Probabilistic Model

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Belief Propagation

How should we treat the calculation of the summation over $2^N$ configuration?

$$\sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_N=0,1} W(x_1, x_2, \cdots, x_N)$$

It is very hard to calculate exactly except some special cases.

- Formulation for approximate algorithm
- Accuracy of the approximate algorithm
Tractable Model

Probabilistic models with no loop are tractable.

\[ \sum_{a} \sum_{b} \sum_{c} \sum_{d} A(a, x)B(b, x)C(c, x)D(d, x) \]

\[ = \left( \sum_{a} A(a, x) \right) \left( \sum_{b} B(b, x) \right) \left( \sum_{c} C(c, x) \right) \left( \sum_{d} D(d, x) \right) \]

Probabilistic models with loop are not tractable.

\[ \sum_{a} \sum_{b} \sum_{c} \sum_{d} W(a, b, c, d, x) \]
Probabilistic model on a graph with no loop

\[ P(a, b, c, d, x, y) \]
\[ \equiv W_A(a, x)W_B(b, x)W_{12}(x, y)W_C(c, y)W_D(d, y) \]

\[ P_2(y) \equiv \sum_a \sum_b \sum_c \sum_d \sum_x P(a, b, c, d, x, y) \]

Marginal probability of the node 2
Probabilistic model on a graph with no loop

\[ P_2(y) = M_{5\rightarrow 2}(y)M_{6\rightarrow 2}(y)M_{1\rightarrow 2}(y) \]

\[ M_{1\rightarrow 2}(y) = \sum_x W_{12}(x, y)M_{3\rightarrow 1}(x)M_{4\rightarrow 1}(x) \]

\[ M_{2\rightarrow 1}(x) = \sum_y W_{12}(x, y)M_{5\rightarrow 2}(y)M_{6\rightarrow 2}(y) \]

Marginal probability can be expressed in terms of the product of messages from all the neighbouring nodes of node 2.

Message from the node 1 to the node 2 can be expressed in terms of the product of message from all the neighbouring nodes of the node 1 except one from the node 2.
Probabilistic Model on a Graph with Loops

\[ P(x_1, x_2, \cdots, x_L) = \frac{1}{Z} \prod_{ij \in N} W_{ij}(x_i, x_j) \]

\[ Z \equiv \sum_{\Omega} \prod_{x \mid ij \in N} W_{ij}(x_i, x_j) \]

Marginal Probability

\[ P_1(x_1) = \sum_{x \setminus \{x_1\}} P(x_1, x_2, \cdots, x_L) \]

\[ P_{12}(x_1, x_2) = \sum_{x \setminus \{x_1, x_2\}} P(x_1, x_2, \cdots, x_L) \]
Belief Propagation

\[ Q_1(\xi) = \sum_\zeta Q_{12}(\xi, \zeta) \]

Message Update Rule

\[ M_{1\to2}(\xi) \propto \sum_\zeta W_{12}(\zeta, \xi) M_{3\to1}(\zeta) \times M_{4\to1}(\zeta) M_{5\to1}(\zeta) \]

\[ Q_1(x_1) = \frac{1}{Z_1} M_{2\to1}(x_1) M_{3\to1}(x_1) \times M_{4\to1}(x_1) M_{5\to1}(x_1) \]

\[ Q_{12}(x_1, x_2) = \frac{1}{Z_{12}} M_{3\to1}(x_1) M_{4\to1}(x_1) M_{5\to1}(x_1) \times W_{12}(x_1, x_2) M_{6\to2}(x_2) M_{7\to2}(x_2) M_{8\to2}(x_2) \]
Message Passing Rule of Belief Propagation

\[
M_{1 \rightarrow 2}(\xi) = \frac{\sum W_{12}(\xi', \xi) M_{3 \rightarrow 1}(\xi) M_{4 \rightarrow 1}(\xi) M_{5 \rightarrow 1}(\xi)}{\sum \sum \sum W_{12}(\xi', \xi) M_{3 \rightarrow 1}(\xi) M_{4 \rightarrow 1}(\xi) M_{5 \rightarrow 1}(\xi)}
\]

Fixed Point Equations for Massage

\[
\hat{M} = \Phi(\hat{M})
\]
Fixed Point Equation and Iterative Method

**Fixed Point Equation**

\[ \vec{M}^* = \Phi(\vec{M}^*) \]

**Iterative Method**

\[
\begin{align*}
\vec{M}_1 & \leftarrow \Phi(\vec{M}_0) \\
\vec{M}_2 & \leftarrow \Phi(\vec{M}_1) \\
\vec{M}_3 & \leftarrow \Phi(\vec{M}_2) \\
\vdots
\end{align*}
\]
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Bayesian Image Analysis

Original Image

Degraded Image

Transmission

Noise

\[
\text{Pr}\{\text{Original Image}\|\text{Degraded Image}\} = \frac{\text{Pr}\{\text{Degraded Image}\|\text{Original Image}\}\text{Pr}\{\text{Original Image}\}}{\text{Pr}\{\text{Degraded Image}\}}
\]

Degradation Process

A Priori Probability

A Posteriori Probability

Marginal Likelihood
Bayesian Image Analysis

Degradation Process

\[ g_i = f_i + n_i \]
\[ n_i \sim N(0, \sigma^2) \]

\[ P(g|f, \sigma) = \prod_{i \in \Omega} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2}(f_i - g_i)^2\right) \]
Bayesian Image Analysis

A Priori Probability

\[ f_i, g_j \in (-\infty, +\infty) \]

\[
P(f|\alpha) = \frac{1}{Z_{PR}(\alpha)} \prod_{ij \in N} \exp \left( -\frac{1}{2} \alpha (f_i - f_j)^2 \right)
\]

Generate

Standard Images

Similar?
Bayesian Image Analysis

A Posteriori Probability

\[ P(f | g, \alpha, \sigma) = \frac{P(g | f, \sigma)P(f | \alpha)}{P(g | \alpha, \sigma)} = \frac{1}{Z_{\text{POS}}(g, \alpha, \sigma)} \times \exp \left( -\frac{1}{2\sigma^2} \sum_{i \in \Omega} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{ij \in N} (f_i - f_j)^2 \right) \]

Gaussian Graphical Model

\[ f_i, g_j \in (-\infty, +\infty) \]
Bayesian Image Analysis

\[ P(f \mid \alpha) \rightarrow f \rightarrow P(g \mid f, \sigma) \rightarrow g \]

- **A Priori Probability**
- **Original Image** \( f = \{f_i \mid i \in \Omega \} \)
- **Degraded Image** \( g = \{g_i \mid i \in \Omega \} \)

\[ P(f \mid g, \alpha, \sigma) = \frac{P(g \mid f, \sigma)P(f \mid \alpha)}{P(g \mid \alpha, \sigma)} \]

\[ \hat{f}_i = \int f_i P(f \mid g, \alpha, \sigma) df = \int_{-\infty}^{+\infty} f_i P(f_i \mid g, \alpha, \sigma) df_i \]
Hyperparameter Determination by Maximization of Marginal Likelihood

\[(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(g | \alpha, \sigma)\]

\[f_i = \int f_i P(f | g, \hat{\alpha}, \hat{\sigma}) \, df\]

\[P(g | \alpha, \sigma) = \int P(f, g | \alpha, \sigma) \, df = \int P(g | f, \sigma) P(f | \alpha) \, df\]

\[P(f, g | \alpha, \sigma) = P(g | f, \sigma) P(f | \alpha)\]

\[P(f | \alpha) \rightarrow f \rightarrow P(g | f, \sigma) \rightarrow g\]

\[\downarrow \text{Marginalization}\]

\[P(g | \alpha, \sigma) \rightarrow g\]

Original Image
\[f = \{f_i | i \in \Omega\}\]

Degraded Image
\[g = \{g_i | i \in \Omega\}\]

Marginal Likelihood
Maximization of Marginal Likelihood by EM (Expectation Maximization) Algorithm

Marginal Likelihood

\[ P(g|\alpha, \sigma) = \int P(g|f, \sigma)P(f|\alpha)df \]

\[ Q(\alpha', \sigma'|\alpha, \sigma, g) = \int P(f|g, \alpha, \sigma)\ln P(f, g|\alpha', \sigma')df \]

\[ \frac{\partial}{\partial \sigma} P(g|\alpha, \sigma) = 0, \quad \frac{\partial}{\partial \alpha} P(g|\alpha, \sigma) = 0 \]

Incomplete Data

\[ g = \{g_i | i \in \Omega\} \]

Equivalent

\[ \left[ \frac{\partial}{\partial \alpha'} Q(\alpha', \sigma'|\alpha, \sigma, g) \right]_{\alpha' = \alpha, \sigma' = \sigma} = 0, \quad \left[ \frac{\partial}{\partial \sigma'} Q(\alpha', \sigma'|\alpha, \sigma, g) \right]_{\alpha' = \alpha, \sigma' = \sigma} = 0 \]
Maximization of Marginal Likelihood by EM (Expectation Maximization) Algorithm

Marginal Likelihood
\[ P(g|\alpha, \sigma) = \int P(g|f, \sigma)P(f|\alpha)df \]

Q-Function
\[ Q(\alpha', \sigma'|\alpha, \sigma, g) = \int P(f|g, \alpha, \sigma)\ln P(f, g|\alpha', \sigma')df \]

EM Algorithm
Iterate the following EM-steps until convergence:

E - Step: \[ Q(\alpha', \sigma'|\alpha(t), \sigma(t)) \leftarrow \int P(f|g, \alpha(t), \sigma(t))\ln P(f, g|\alpha', \sigma')df \].

M - Step: \[ (\alpha(t+1), \sigma(t+1)) \leftarrow \arg\max_{(\alpha', \sigma')} Q(\alpha', \sigma'|\alpha(t), \sigma(t)) \].

One-Dimensional Signal

**EM Algorithm**

Original Signal

\[ f_i \]

Degraded Signal

\[ g_i \]

\[ \sigma = 40 \]

Estimated Signal

\[ \hat{f}_i \]

\[ (\hat{\sigma}, \hat{\alpha}) = (31.25, 0.0217) \]
Image Restoration by Gaussian Graphical Model

Original Image  Degraded Image

![Original Image](image1.png)  ![Degraded Image](image2.png)

Original Image: MSE: 1512
Degraded Image: MSE: 1529

EM Algorithm with Belief Propagation

![Graph](graph.png)
Exact Results of Gaussian Graphical Model

\[ P(f|g, \alpha, \sigma) = \frac{1}{Z_{\text{POS}}(g, \alpha, \sigma)} \exp \left( -\frac{1}{2\sigma^2} \sum_{i \in \Omega} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{i,j \in N} (f_i - f_j)^2 \right) \]

\[ = \frac{\exp \left( -\frac{1}{2\sigma^2} \|f - g\|^2 - \frac{1}{2} \alpha f^T Cf \right)}{\int \exp \left( -\frac{1}{2\sigma^2} \|f - g\|^2 - \frac{1}{2} \alpha f^T Cf \right) df} \]

Multi-dimensional Gauss integral formula

\[ P(g|\alpha, \sigma) = \frac{\det(\alpha C)}{\sqrt{(2\pi)^{|\Omega|} \det(I + \alpha \sigma^2 C)}} \exp \left( -\frac{1}{2} \alpha g^T \frac{C}{I + \alpha \sigma^2 C} g \right) \]

\[ (\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(g|\alpha, \sigma) \]

\[ \hat{f} = \left( I + \alpha \sigma^{-2} C \right)^{-1} g \]
**Comparison of Belief Propagation with Exact Results in Gaussian Graphical Model**

\[(\hat{\alpha}, \hat{\sigma}) = \text{arg max } P(g|\alpha, \sigma)\]

|                | MSE   | \(\hat{\alpha}\) | \(\hat{\sigma}\) | \(\ln P(g|\hat{\alpha}, \hat{\sigma})\) |
|----------------|-------|-------------------|-------------------|-----------------------------------------|
| **Belief Propagation** |       |                   |                   |                                         |
|                | 327   | 0.000611          | 36.302            | -5.19201                                |
| **Exact**      | 315   | 0.000759          | 37.919            | -5.21444                                |
| **Belief Propagation** |       |                   |                   |                                         |
|                | 260   | 0.000574          | 33.998            | -5.15241                                |
| **Exact**      | 236   | 0.000652          | 34.975            | -5.17528                                |

\[\text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2\]
Image Restoration by Gaussian Graphical Model

Original Image

Degraded Image

Belief Propagation

Exact

MSE: 1512

MSE: 325

MSE: 315

Lowpass Filter

Wiener Filter

Median Filter

MSE: 411

MSE: 545

MSE: 447

\[
\text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2
\]
Image Restoration by Gaussian Graphical Model

Original Image  Degraded Image  Belief Propagation  Exact

Lowpass Filter  Wiener Filter  Median Filter

MSE: 1529  MSE: 260  MSE 236

MSE: 224  MSE: 372  MSE: 244

$$\text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2$$
Extension of Belief Propagation

**Generalized Belief Propagation**


**Generalized belief propagation is equivalent to the cluster variation method in statistical mechanics**

Image Restoration by Gaussian Graphical Model

\(\sigma = 40\)

\[
\text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2
\]

\((\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(g|\alpha, \sigma)\)

| Method                          | MSE  | \(\hat{\alpha}\) | \(\hat{\sigma}\) | \(\ln P(g|\hat{\alpha}, \hat{\sigma})\) |
|---------------------------------|------|-------------------|-------------------|----------------------------------------|
| Belief Propagation              | 327  | 0.000611          | 36.302            | -5.19201                               |
| Generalized Belief Propagation  | 315  | 0.000758          | 37.909            | -5.21172                               |
| Exact                           | 315  | 0.000759          | 37.919            | -5.21444                               |

\(\sigma = 40\)

| Method                          | MSE  | \(\hat{\alpha}\) | \(\hat{\sigma}\) | \(\ln P(g|\hat{\alpha}, \hat{\sigma})\) |
|---------------------------------|------|-------------------|-------------------|----------------------------------------|
| Belief Propagation              | 260  | 0.000574          | 33.998            | -5.15241                               |
| Generalized Belief Propagation  | 236  | 0.000652          | 34.971            | -5.17256                               |
| Exact                           | 236  | 0.000652          | 34.975            | -5.17528                               |
Image Restoration by Gaussian Graphical Model and Conventional Filters

\[ \sigma = 40 \]

\[
MSE = \frac{1}{|\Omega|} \sum_{(x,y) \in \Omega} (f_{x,y} - \hat{f}_{x,y})^2
\]

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<th>MSE</th>
<th>Lowpass Filter</th>
<th>MSE</th>
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<td>(5x5) 413</td>
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<tr>
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<td>(3x3) 486</td>
<td>(5x5) 445</td>
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<tr>
<td><strong>Exact</strong></td>
<td>315</td>
<td>(3x3) 864</td>
<td>(5x5) 548</td>
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(3x3) Lowpass  (5x5) Median  (5x5) Wiener
Image Restoration by Gaussian Graphical Model and Conventional Filters

$\sigma = 40$

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$MSE = \frac{1}{|\Omega|} \sum_{(x,y) \in \Omega} (f_{x,y} - \hat{f}_{x,y})^2$

(5x5) Lowpass (5x5) Median (5x5) Wiener
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Summary

- Formulation of belief propagation
- Accuracy of belief propagation in Bayesian image analysis by means of Gaussian graphical model (Comparison between the belief propagation and exact calculation)