Statistical performance analysis by loopy belief propagation in probabilistic image processing

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Introduction

Bayesian network is originally one of the methods for probabilistic inferences in artificial intelligence. Some probabilistic models for information processing are also regarded as Bayesian networks. Bayesian networks are expressed in terms of products of functions with a couple of random variables and can be associated with graphical representations. Such probabilistic models for Bayesian networks are referred to as Graphical Model.
Probabilistic image processing systems are formulated on square grid graphs. Averages, variances and covariances of the Bayesian network are approximately computed by using the belief propagation on the square grid graph.
MRF, Belief Propagation and Statistical Performance

Geman and Geman (1986): IEEE Transactions on PAMI
Image Processing by Markov Random Fields (MRF)

Cluster Variation Method for MRF in Image Processing
CVM = Generalized Belief Propagation (GBP)

Statistical Performance Estimation for MRF
(Infinite Range Ising Model and Replica Theory)

Is it possible to estimate the performance of belief propagation statistically?
Outline

1. Introduction
2. Bayesian Image Analysis by Gauss Markov Random Fields
3. Statistical Performance Analysis for Gauss Markov Random Fields
4. Statistical Performance Analysis in Binary Markov Random Fields by Loopy Belief Propagation
5. Concluding Remarks
Outline

1. Introduction
2. Bayesian Image Analysis by Gauss Markov Random Fields
3. Statistical Performance Analysis for Gauss Markov Random Fields
4. Statistical Performance Analysis in Binary Markov Random Fields by Loopy Belief Propagation
5. Concluding Remarks
Image Restoration by Bayesian Statistics
Image Restoration by Bayesian Statistics

Original Image

Noise

Transmission

Degraded Image
Image Restoration by Bayesian Statistics
Image Restoration by Bayesian Statistics

\[
\text{Bayes Formula}
\]

\[
\Pr\{\text{Original Image} | \text{Degraded Image}\} \propto \Pr\{\text{Degraded Image} | \text{Original Image}\} \Pr\{\text{Original Image}\}
\]

Transmission

Noise

Posterior

Original Image

Degraded Image

Estimate
Image Restoration by Bayesian Statistics

Bayes Formula

\[ \Pr\{\text{Original Image} \mid \text{Degraded Image}\} \]

\[ \propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\} \Pr\{\text{Original Image}\} \]

Assumption 1: Original images are randomly generated by according to a prior probability.
Image Restoration by Bayesian Statistics

**Assumption 2:** Degraded images are randomly generated from the original image by according to a conditional probability of degradation process.

**Bayes Formula**

\[
\Pr\{\text{Original Image} \mid \text{Degraded Image}\} \propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\} \times \Pr\{\text{Original Image}\}
\]
Bayesian Image Analysis

Assumption: Prior Probability consists of a product of functions defined on the neighbouring pixels.

Prior Probability

\[ \Pr\{\vec{F} = \vec{f} \mid \alpha\} \propto \prod_{\{i,j\} \in E} \exp\left(-\frac{1}{2} \alpha (f_i - f_j)^2 \right) = \exp\left(-\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2 \right) \]

\(
\Pr\{\text{Original Image} \mid \text{Degraded Image}\}
\)

\(\propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\}\)

Prior

\(\times \Pr\{\text{Original Image}\}\)

Posterior
Bayesian Image Analysis

Assumption: Prior Probability consists of a product of functions defined on the neighbouring pixels.

Prior Probability

\[
\Pr\{\tilde{F} = \tilde{f} | \alpha\} \propto \prod_{\{i,j\} \in E} \exp\left(-\frac{1}{2} \alpha (f_i - f_j)^2\right) = \exp\left(-\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2\right)
\]

\(\alpha > 0\)

\(V = \{1,2,3,4,5,6,7,8,9,10,11,12\} : \text{Set of all the pixels}\)

\(E = \{1,2\}, \{2,3\}, \{3,4\}, \{5,6\}, \{6,7\}, \{7,8\}, \{9,10\}, \{10,11\}, \{10,11\}, \{11,12\},\)

\{11,12\}, \{1,5\}, \{2,6\}, \{3,7\}, \{4,8\}, \{5,9\}, \{6,10\}, \{7,11\}, \{8,12\}\)

Set of all the neighbouring pairs of pixels

Random Field of Original Image \(\tilde{F} = (F_1, F_2, \ldots, F_{12})\)

\(\tilde{f}_{i,j} = \exp\left(-\frac{1}{2} \alpha (F_i - F_j)^2\right)\)

\(i : \text{Random Variable } F_i \text{ of } i\)-th pixel in Original Image

Posterior

\[
\Pr\{\text{Original Image} | \text{Degraded Image}\}
\]

Likelihood

\[
\propto \Pr\{\text{Degraded Image} | \text{Original Image}\}
\]

Prior

\[
\times \Pr\{\text{Original Image}\}
\]
Bayesian Image Analysis

Assumption: Prior Probability consists of a product of functions defined on the neighbouring pixels.

Prior Probability

\[
\Pr\{\vec{F} = \vec{f} \mid \alpha\} \propto \prod_{\{i,j\} \in E} \exp\left(-\frac{1}{2} \alpha (f_i - f_j)^2\right) = \exp\left(-\frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2\right)
\]

\[
\alpha > 0
\]

Patterns by MCMC.

\[
\alpha = 0.0001 \quad \alpha = 0.0005 \quad \alpha = 0.0030
\]

14 October, 2010
LRI Seminar 2010 (Univ. Paris-Sud)
Bayesian Image Analysis

Assumption: Degraded image is generated from the original image by Additive White Gaussian Noise.

\[
\Pr\{\tilde{G} = \tilde{g} \mid \tilde{F} = \tilde{f}, \sigma\} \propto \prod_{i \in V} \exp\left( -\frac{1}{2\sigma^2} (g_i - f_i)^2 \right)
\]

\(V: \text{Set of all the pixels} \quad \sigma > 0\)
Bayesian Image Analysis

Assumption: Degraded image is generated from the original image by Additive White Gaussian Noise.

\[
\Pr\{\tilde{G} = \tilde{g} \mid \tilde{F} = \tilde{f}, \sigma\} \\
\propto \prod_{i \in V} \exp\left(-\frac{1}{2\sigma^2}(g_i - f_i)^2\right)
\]

\(V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} : \) Set of all the pixels

\(i : \) Random Variable \(F_i\) of \(i\)-th pixel in Original Image

\(i : \) Random Variable \(G_i\) of \(i\)-th pixel in Degraded Image

\[
\exp\left(-\frac{1}{2\sigma^2}(G_i - F_i)^2\right)
\]

\(\tilde{F} = (F_1, F_2, \ldots, F_{12}) : \) Random Field of Original Image

\(\tilde{G} = (G_1, G_2, \ldots, G_{12}) : \) Random Field of Degraded Image
Bayesian Image Analysis

Assumption: Degraded image is generated from the original image by Additive White Gaussian Noise.

\[
\text{Posterior} \quad \Pr\{\text{Original Image} \mid \text{Degraded Image}\}
\]

\[
\text{Likelihood} \quad \propto \Pr\{\text{Degraded Image} \mid \text{Original Image}\}
\]

\[
\text{Prior} \quad \propto \Pr\{\text{Original Image}\}
\]

\[
\prod_{i \in V} \frac{\Pr F_i}{\Pr G_i} \propto \prod_{i \in V} \exp\left(-\frac{1}{2\sigma^2}(g_i - f_i)^2\right)
\]

\[
\bar{F} = (F_1, F_2, \ldots, F_{12}) : \text{Random Field of Original Image}
\]

\[
\bar{G} = (G_1, G_2, \ldots, G_{12}) : \text{Random Field of Degraded Image}
\]

\[
\frac{\Pr G = \bar{g} \mid \bar{F} = \bar{f}, \sigma}{\Pr G = \bar{g} \mid \bar{F} = \bar{f}} \propto \prod_{i \in V} \exp\left(-\frac{1}{2\sigma^2}(g_i - f_i)^2\right)
\]

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Bayesian Image Analysis

Prior Probability

\[ \Pr\{\tilde{F} = f | \alpha\} \]

Original Image

\[ f \]

Degradation Process

\[ \Pr\{\tilde{G} = \tilde{g} | \tilde{F} = \tilde{f}, \sigma\} \]

Degraded Image

\[ \tilde{g} \]

Posterior Probability

\[ \Pr\{\tilde{F} = \tilde{f} | \tilde{G} = \tilde{g}, \alpha, \sigma\} = \frac{\Pr\{\tilde{G} = \tilde{g} | \tilde{F} = \tilde{f}, \sigma\} \Pr\{\tilde{F} = \tilde{f}, \alpha\}}{\Pr\{\tilde{G} = \tilde{g} | \alpha, \sigma\}} \]

\[ \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i,j\} \in E} (f_i - f_j)^2\right) \]
Bayesian Image Analysis

Prior Probability

Original Image

Degradation Process

Degraded Image

Posterior Probability

Estimate

Data Dominant

Smoothing

Pr\{\tilde{F} = \tilde{f} | \alpha\}

Pr\{\tilde{G} = \tilde{g} | \tilde{F} = \tilde{f}, \sigma\}

\Pr\{\tilde{F} = \tilde{f} | \tilde{G} = \tilde{g}, \alpha, \sigma\} = \frac{\Pr\{\tilde{G} = \tilde{g} | \tilde{F} = \tilde{f}, \sigma\} \Pr\{\tilde{F} = \tilde{f}, \alpha\}}{\Pr\{\tilde{G} = \tilde{g} | \alpha, \sigma\}}

\propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \frac{\alpha}{\{i, j\} \in E} (f_i - f_j)^2\right)
Bayesian Image Analysis

Prior Probability

$\Pr\{\tilde{F} = \tilde{f} \mid \alpha\}$

Original Image

Degradation Process

$\Pr\{\tilde{G} = \tilde{g} \mid \tilde{F} = \tilde{f}, \sigma\}$

Degraded Image

Posterior Probability

Data Dominant

Smoothing

Bayesian Network

Estimate

$\Pr\{\tilde{F} = \tilde{f} \mid \tilde{G} = \tilde{g}, \alpha, \sigma\}$

$= \Pr\{\tilde{G} = \tilde{g} \mid \tilde{F} = \tilde{f}, \sigma\} \Pr\{\tilde{F} = \tilde{f} \mid \alpha\}$

$\propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2\right)$

$i, j =$ Random Variable $F_i$ of $i$-th pixel in Original Image

$i, j =$ Random Variable $G_i$ of $i$-th pixel in Degraded Image

$\tilde{F} = (F_1, F_2, \ldots, F_{12})$
Random Field of Original Image

$\tilde{G} = (G_1, G_2, \ldots, G_{12})$
Random Field of Degraded Image
Bayesian Image Analysis

Prior Probability

\[ \Pr\{ \vec{F} = \vec{f} | \alpha \} \]

Original Image

Degradation Process

\[ \Pr\{ \vec{G} = \vec{g} | \vec{F} = \vec{f}, \sigma \} \]

Degraded Image

Posterior Probability

\[
\begin{align*}
\Pr\{ \vec{F} = \vec{f} | \vec{G} = \vec{g}, \alpha, \sigma \} &= \Pr\{ \vec{G} = \vec{g} | \vec{F} = \vec{f}, \sigma \} \Pr\{ \vec{F} = \vec{f}, \alpha \} \\
&\propto \exp\left( -\frac{1}{2\sigma^2} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right)
\end{align*}
\]

Estimate

\[ \hat{f} = h(\vec{g}, \alpha, \sigma) \]

\[ = \int \vec{z} \Pr\{ \vec{F} = \vec{z} | \vec{G} = \vec{g}, \alpha, \sigma \} d\vec{z} \]

\[ = (I + \alpha \sigma^2 C)^{-1} \vec{g} \]

\[ \langle i | c | j \rangle = \begin{cases} \delta_{ij}, & i = j \in V \\ -1, & \{i, j\} \in E \\ 0, & \text{otherwise} \end{cases} \]

Bayesian Network

\[ = \left( \prod_{i \in V} F_i \right) \left( \prod_{\{i, j\} \in E} F_i \right) \]

\[ F_i \in (\infty, +\infty) \]

⇒ Gauss Markov Random Field

14 October, 2010

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Image Restorations by Gaussian Markov Random Fields and Conventional Filters

\[ \text{MSE} = \frac{1}{|V|} \sum_{i \in V} \| f - \hat{f} \|^2 \]

\( V \): Set of all the pixels

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Size</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss Markov Random Field</td>
<td></td>
<td>315</td>
</tr>
<tr>
<td>Lowpass Filter</td>
<td>(3x3)</td>
<td>388</td>
</tr>
<tr>
<td></td>
<td>(5x5)</td>
<td>413</td>
</tr>
<tr>
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Outline

1. Introduction
2. Bayesian Image Analysis by Gauss Markov Random Fields
3. Statistical Performance Analysis for Gauss Markov Random Fields
4. Statistical Performance Analysis in Binary Markov Random Fields by Loopy Belief Propagation
5. Concluding Remarks
Statistical Performance
by Sample Average of Numerical Experiments
Statistical Performance by Sample Average of Numerical Experiments

\[
Pr\{G|F=f, \sigma\}
\]

Original Images

\[
\begin{align*}
\vec{g}_1 \\
\vec{g}_2 \\
\vec{g}_3 \\
\vec{g}_4 \\
\vec{g}_5
\end{align*}
\]

Noise

Observed Data
Statistical Performance by Sample Average of Numerical Experiments

Original Images

Noise

Pr\{G|F=f,σ\}

Posterior Probability

Pr\{F|G=g,α,σ\}

Observed Data

Estimated Results
Statistical Performance by Sample Average of Numerical Experiments

Sample Average of Mean Square Error

\[ D(\alpha, \sigma \mid \vec{f}, \sigma) \approx \frac{1}{5} \sum_{n=1}^{5} \| \vec{f} - \vec{h}_n \|^2 \]

Original Images

Pr\{G|\vec{F}=f, \sigma\}

Noise

G

Pr\{G=g, \alpha, \sigma\}

Posterior Probability

\[ \sum \approx 5 \]
Statistical Performance Estimation

\[ D(\alpha, \sigma \mid \tilde{f}, \sigma) = \frac{1}{|V|} \int \left\| \tilde{h}(\tilde{g}, \alpha, \sigma) - \tilde{f} \right\|^2 d\tilde{g} \]

Pr{\tilde{G} = \tilde{g} \mid \tilde{F} = \tilde{f}, \sigma}

Original Image

\[ \tilde{G} = \tilde{g}, \alpha, \sigma \]

Posterior Probability

Additive White Gaussian Noise

Restored Image

Pr{\tilde{F} = \tilde{f} \mid \tilde{G} = \tilde{g}, \alpha, \sigma}

\[ \tilde{h}(\tilde{g}, \alpha, \sigma) = \left( I + \alpha \sigma^2 C \right)^{-1} \tilde{g} \]

Additive White Gaussian Noise

\[ \tilde{F} = (F_1, F_2, \ldots, F_{12}) \]

\[ \tilde{G} = (G_1, G_2, \ldots, G_{12}) \]

\[ \langle i | C | j \rangle = \begin{cases} |\partial_i|, & i = j \in V \\ -1, & \{i, j\} \in E \\ 0, & \text{otherwise} \end{cases} \]
Statistical Performance Estimation for Gauss Markov Random Fields

\[ D(\alpha, \sigma \mid \bar{f}, \sigma) = \frac{1}{|V|} \int \left\| \bar{f} - \bar{h}(\bar{g}, \alpha, \sigma) \right\|^2 \Pr\{\bar{G} = \bar{g} \mid \bar{F} = \bar{f}, \sigma\} \, d\bar{g} \]

\[ = \frac{1}{|V|} \int \left\| \bar{f} - \frac{I}{I + \alpha \sigma^2 C} \bar{g} \right\|^2 \left( \frac{1}{\sqrt{2\pi \sigma}} \right)^{|V|} \exp\left(-\frac{1}{2\sigma^2} \left\| \bar{g} - \bar{f} \right\|^2 \right) \, d\bar{g} \]

\[ = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 I}{(I + \alpha \sigma^2 C)^2} \right) + \frac{1}{|V|} \bar{f}^T \frac{\alpha^2 \sigma^4 C^2}{(I + \alpha \sigma^2 C)^2} \bar{f} \]

\[ = \frac{1}{|V|} \text{Tr} \left( \frac{\sigma^2 I}{(I + \alpha \sigma^2 C)^2} \right) \]

\[ | \bar{i} |, \quad i = j \in V \]

\[ -1, \quad \{i, j\} \in E \]

\[ 0, \quad \text{otherwise} \]
Outline

1. Introduction
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3. Statistical Performance Analysis for Gauss Markov Random Fields
4. Statistical Performance Analysis in Binary Markov Random Fields by Loopy Belief Propagation
5. Concluding Remarks
In order to compute the marginal probability \( \Pr\{F_2 | G = \tilde{g}\} \), we take summations over all the pixels except the pixel 2.
Marginal Probability in Belief Propagation

\[
\Pr\left\{ F_2 \mid \bar{G} = \bar{g} \right\} = \sum_{F_1} \sum_{F_3} \sum_{F_4} \cdots \sum_{F_N} \Pr\left\{ F_1, F_2, F_3, F_4, \cdots, F_N \mid \bar{G} = \bar{g} \right\}
\]

\[
= \sum_{F_1} \sum_{F_3} \sum_{F_4} \cdots \sum_{F_N}
\]

Diagram of a belief propagation network with nodes and edges.
Marginal Probability in Belief Propagation

$$\Pr\{F_2 \mid \vec{G} = \vec{g}\} = \sum_{F_1} \sum_{F_3} \sum_{F_4} \cdots \sum_{F_N} \Pr\{F_1, F_2, F_3, F_4, \ldots, F_N \mid \vec{G} = \vec{g}\}$$

$$= \sum_{F_1} \sum_{F_3} \sum_{F_4} \cdots \sum_{F_N}$$

In the belief propagation, the marginal probability $\Pr\{F_2 \mid G=g\}$ is approximately expressed in terms of the messages from the neighbouring region of the pixel 2.
Marginal Probability in Belief Propagation

\[
\Pr\{F_1, F_2 \mid \vec{G} = \vec{g}\} = \sum_{F_3} \sum_{F_4} \cdots \sum_{F_N} \Pr\{F_1, F_2, F_3, F_4, \ldots, F_N \mid \vec{G} = \vec{g}\}
\]

= \sum_{F_3} \sum_{F_4} \cdots \sum_{F_N} \Pr\{F_1, F_2 \mid \vec{G} = \vec{g}\}

In order to compute the marginal probability \( \Pr\{F_1, F_2 \mid G=g\} \), we take summations over all the pixels except the pixels 1 and 2.
In the belief propagation, the marginal probability $\text{Pr}\{F_1, F_2 | \tilde{G} = \tilde{g}\}$ is approximately expressed in terms of the messages from the neighbouring region of the pixels 1 and 2.
Belief Propagation in Probabilistic Image Processing

Belief Propagation in Probabilistic Image Processing

Pr\{F_1,F_2, \ldots, F_N | \mathcal{G} = \mathcal{g} \} = \text{exp}\left(-\frac{1}{2\sigma^2}(F_i - G_i)^2\right)

Pr\{F_i | \mathcal{G} = \mathcal{g} \} = \text{exp}\left(-\frac{1}{2}\alpha(F_i - F_j)^2\right)

\mathcal{M}_j \rightarrow_i (F_i)

Approximate representation of marginal probability for a single pixel

\text{Approximate representation of marginal probability for neighbouring pixels}

Pr\{F_2 | \mathcal{G} = \mathcal{g} \} = \sum_{F_1} \Pr\{F_1,F_2 | \mathcal{G} = \mathcal{g} \}

\sum_{F_2} \sum_{F_1,F_2} \Pr\{F_1,F_2 | \mathcal{G} = \mathcal{g} \}

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Belief Propagation in Probabilistic Image Processing

\[
\prod_{i \in V} i \prod_{\{i,j\} \in E} i \rightarrow j
\]

\(i\) : Random Variable \(F_i\) of \(i\)-th pixel in Original Image
\(j\) : Random Variable \(G_i\) of \(i\)-th pixel in Degraded Image

\[
i \rightarrow j = \exp\left(-\frac{1}{2\sigma^2}(F_i - F_j)^2\right)
\]

\(\vec{F} = (F_1, F_2, \ldots, F_{12})\)
Random Field of Original Image

\(\vec{G} = (G_1, G_2, \ldots, G_{12})\)
Random Field of Degraded Image

\[
\sum_{F_i} \sum_{F_j} i \rightarrow j = \sum_{F_i} \sum_{F_j} i \rightarrow j
\]

\((\forall \{i,j\} \in E)\) converge

\[
\Pr\{F_i | \vec{G} = \vec{g}\} = \frac{\sum_{F_i} \Pr\{F_i,F_j | \vec{G} = \vec{g}\}}{\sum_{F_i} \Pr\{F_i,F_j | \vec{G} = \vec{g}\}}
\]

\((\forall i \in V)\)
**Image Restorations by Gaussian Markov Random Fields and Conventional Filters**

<table>
<thead>
<tr>
<th></th>
<th>MSE</th>
</tr>
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<tbody>
<tr>
<td>Gauss MRF (Exact)</td>
<td>315</td>
</tr>
<tr>
<td>Gauss MRF (Belief Propagation)</td>
<td>327</td>
</tr>
<tr>
<td>Lowpass Filter</td>
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<td>445</td>
</tr>
</tbody>
</table>

\[
\text{MSE} = \frac{1}{|V|} \sum_{i \in V} (f_i - \hat{f}_i)^2
\]

\(V\): Set of all the pixels

\(F_i \in (-\infty, +\infty)\)

\(\Rightarrow\) Gauss Markov Random Field
Gray-Level Image Restoration (Spike Noise)

Degraded Image

Belief Propagation

MSE: 2075

MSE: 244

MSE: 371

MSE: 3469

MSE: 217

MSE: 523

MSE: 2075

MSE: 395

MSE: 135

Original Image

Lowpass Filter

Median Filter
Binary Image Restoration by Loopy Belief Propagation

\[ \alpha^* = 0.465 \]

\[ \Pr\{F|\alpha = 0.465\} \]

\[ \hat{\alpha}_{LBP} = 0.4343 \]

Loopy Belief Propagation

\[ \hat{\alpha}_{LBP} = \arg\max_{\alpha} \Pr\{F=f|\alpha\} \]

\[ \hat{\alpha}_{LBP} = 0.4171 \]

\[ \hat{\alpha}_{LBP} = 0.3991 \]
Statistical Performance by Sample Average of Numerical Experiments

Sample Average of Mean Square Error

\[ D(\alpha, \sigma | \vec{f}, \sigma) \approx \frac{1}{5} \sum_{n=1}^{5} \| \vec{f} - \vec{h}_n \|^2 \]

Original Images

Noise \( \Pr\{G|F=f, \sigma\} \)

Posterior Probability \( \Pr\{F|G=g, \alpha, \sigma\} \)

Estimated Results

Observed Data
 Statistical Performance Estimation for Binary Markov Random Fields

\[
D(\alpha, \sigma \mid \vec{f}, \sigma) = \frac{1}{|V|} \int \left\| \vec{h}(\vec{g}, \alpha, \sigma) - \vec{f} \right\|^2 \Pr\{\vec{G} = \vec{g} \mid \vec{F} = \vec{f}, \sigma\} \, d\vec{g}
\]

\[
f_i = \pm 1
\]

Light intensities of the original image can be regarded as spin states of ferromagnetic system.

Free Energy of Ising Model with Random External Fields

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Statistical Performance Estimation for Binary Markov Random Fields

\[ D(\alpha, \sigma \mid \bar{f}, \sigma) = \frac{1}{|V|} \int \| \bar{h}(\bar{g}, \alpha, \sigma) - \bar{f} \|^2 \Pr\{\bar{G} = \bar{g} \mid \bar{F} = \bar{f}, \sigma\} \, d\bar{g} \]

\[ \approx \frac{1}{|V|} \sum_{i \in V} \int_{-\infty}^{\infty} dg_i \left( \prod_{j \in \partial i} \int_{-\infty}^{\infty} d\lambda_{j \rightarrow i} \right) (f_i - h_i(\bar{g}, \alpha, \sigma))^2 \left( \prod_{j \in \partial i} \rho_{j \rightarrow i}(\lambda_{j \rightarrow i}) \right) \Pr\{G_i = g_i \mid F_i = f_i, \sigma\} \]

\[ \rho_{j \rightarrow i}(\lambda_{j \rightarrow i}) = \sum_{i \in V} \int_{-\infty}^{\infty} dg_i \left( \prod_{k \in \partial j \setminus \{i\}} \int_{-\infty}^{\infty} d\lambda_{k \rightarrow j} \right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(g_i - \lambda_{j \rightarrow i})^2}{2\sigma^2}} \]

\[ h_i(\bar{g}, \alpha, \sigma) \approx \text{sgn} \left( \frac{g_i}{\sigma^2} + \sum_{j \in \partial i} \lambda_{j \rightarrow i} \right) \]

\[ \times \delta \left[ \lambda_{j \rightarrow i} - \tanh^{-1} \left( \frac{\tanh(\alpha) \tanh \left( \frac{g_i}{\sigma^2} + \sum_{l \in \partial j \setminus \{i\}} \lambda_{l \rightarrow j} \right)}{\sigma} \right) \right] \]

\[ \times \left( \prod_{k \in \partial j \setminus \{i\}} \rho_{k \rightarrow j}(\lambda_{k \rightarrow j}) \right) \Pr\{G_j = g_j \mid F_j = f_j, \sigma^*\} \]
Statistical Performance Estimation for Markov Random Fields

\[ D(\alpha, \sigma | \tilde{f}, \sigma) = \frac{1}{V} \int \| h(\tilde{g}, \alpha, \sigma) - \tilde{f} \|^2 \Pr \{ \tilde{G} = \tilde{g} | \tilde{F} = \tilde{f}, \sigma \} \, d\tilde{g} \]

Spin Glass Theory in Statistical Mechanics
Loopy Belief Propagation

Multi-dimensional Gauss Integral Formulas

14 October, 2010 LRI Seminar 2010 (Univ. Paris-Sud)
Statistical Performance Estimation for Markov Random Fields

\[ D(\alpha, \sigma | \tilde{f}, \sigma) = \frac{1}{|V|} \int \| \tilde{f} - \tilde{h}(\bar{g}, \alpha, \sigma) \|^2 \Pr\{ \tilde{G} = \bar{g} | \tilde{F} = \tilde{f}, \sigma \} \, d\bar{g} \]
Outline

1. Introduction
2. Bayesian Image Analysis by Gauss Markov Random Fields
3. Statistical Performance Analysis for Gauss Markov Random Fields
4. Statistical Performance Analysis in Binary Markov Random Fields by Loopy Belief Propagation
5. Concluding Remarks
Summary

Formulation of probabilistic model for image processing by means of conventional statistical schemes has been summarized.

Statistical performance analysis of probabilistic image processing by using Gauss Markov Random Fields has been shown.

One of extensions of statistical performance estimation to probabilistic image processing with discrete states has been demonstrated.
Our framework can be extended to erase a scribbling.

Gauss MRF and LBP
References


