Introduction to Probabilistic Image Processing and Bayesian Networks

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More is different

Probabilistic information processing can give us unexpected capacity in a system constructed from many cooperating elements with randomness. The circumstances are similar to everything in the natural world being made up of a lot of molecules with interactions and fluctuations. The collection of molecules can give rise to unpredictable phenomena. This is often called “More is different” in physics.
Main Interests
Information Processing:
  Data
  Physics:
    Material,
    Natural Phenomena

Some physical concepts in Physical models are useful for the design of computational models in probabilistic information processing.

System of a lot of elements with mutual relation
Common Concept between Computer Sciences and Physics

Data is constructed from many bits
0,1 → 101101
   110001
   Bit

A sequence is formed by deciding the arrangement of bits.

Materials are constructed from a lot of molecules.

A lot of elements have mutual relation of each other

Molecules have interactions of each other.

29 December, 2008
National Tsing Hua University, Taiwan
More is different in informatics as well.

Our goal is to establish theoretical paradigms for probabilistic information processing by means of statistical science and statistical physics. The probabilistic information processing is based on both modeling of problems and design of algorithms, which is often realized as graphical models including Bayesian network.
Purpose of My Talk

- Review of formulation of probabilistic model for image processing by means of conventional statistical schemes.
- Review of probabilistic image processing by using Gaussian graphical model (Gauss Markov Random Fields) as the most basic example.
- Review of how to construct a belief propagation algorithm for image processing.
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2. **Probabilistic Image Processing**
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Bayes Formula and Bayesian Network

\[ \Pr\{A \mid B\} = \frac{\Pr\{B \mid A\} \Pr\{A\}}{\Pr\{B\}} \]

**Bayes Rule**

Event \( B \) is given as the observed data. Event \( A \) corresponds to the original information to estimate. Thus the Bayes formula can be applied to the estimation of the original information from the given data.
Image Restoration by Probabilistic Model

Assumption 1: The degraded image is randomly generated from the original image by according to the degradation process.
Assumption 2: The original image is randomly generated by according to the prior probability.

\[
\text{Posterior} = \frac{\text{Pr}\{\text{Degraded Image} \mid \text{Original Image}\} \times \text{Pr}\{\text{Original Image}\}}{\text{Marginal Likelihood}}
\]
The original images and degraded images are represented by \( f = (f_1, f_2, \ldots, f_{|V|})^T \) and \( g = (g_1, g_2, \ldots, g_{|V|})^T \), respectively.
Probabilistic Modeling of Image Restoration

Pr\{Original Image | Degraded Image\}

Prior Likelihood

Assumption 1: A given degraded image is obtained from the original image by changing the state of each pixel to another state by the same probability, independently of the other pixels.

\[ P(g_1, g_2, \ldots, g_{|V|} | f_1, f_2, \ldots, f_{|V|}) = \prod_{i \in V} \Psi(g_i | f_i) \]

Random Fields

Posterior

Pr\{Degraded Image | Original Image\}

Pr\{Original Image\}
Probabilistic Modeling of Image Restoration

Posterior
Pr\{Original Image | Degraded Image\}

Likelihood
$\propto Pr\{\text{Degraded Image} | \text{Original Image}\} Pr\{\text{Original Image}\}$

Prior

Assumption 2: The original image is generated according to a prior probability. Prior Probability consists of a product of functions defined on the neighbouring pixels.

$$P(f_1, f_2, \ldots, f_{|V|}) = \prod_{\{i,j\} \in E} \Phi(f_i, f_j)$$

Product over All the Nearest Neighbour Pairs of Pixels
Prior Probability for Binary Image

It is important how we should assume the function $\Phi(f_i, f_j)$ in the prior probability.

$$P(f) = \prod_{\{i, j\} \in E} \Phi(f_i, f_j)$$

We assume that every nearest-neighbour pair of pixels take the same state of each other in the prior probability.

$$\Phi(1,1) = \Phi(0,0) > \Phi(0,1) = \Phi(1,0)$$

It is important how we should assume the function $\Phi(f_i, f_j)$ in the prior probability.

$$f_i = 0, 1$$

Probability of Neighbouring Pixel
Prior Probability for Binary Image

Which state should the center pixel be taken when the states of neighbouring pixels are fixed to the white states?

Prior probability prefers to the configuration with the least number of red lines.
Prior Probability for Binary Image

Which state should the center pixel be taken when the states of neighbouring pixels are fixed as this figure?

Prior probability prefers to the configuration with the least number of red lines.
What happens for the case of large number of pixels?

Covariance between the nearest neighbour pairs of pixels

\[ p \]

Sampling by Marko chain Monte Carlo

Patterns with both ordered states and disordered states are often generated near the critical point.

\[ \text{small } p \rightarrow \ln p \rightarrow \text{large } p \]

Disordered State

Critical Point (Large fluctuation)

Ordered State
Physical model of ferromagnetism and Probabilistic model of image processing

Ising Model

Regular graph

Up Spin State

Down Spin State

Probabilistic models for image processing has the similar structure as physical models of ferromagnetism

Markov Random Field (MRF) Model

Black State

White State
We regard that patterns generated near the critical point are similar to the local patterns in real world images.
Bayesian Image Analysis

Prior Probability

\[ P(f) \]

\( V \): Set of All the pixels

Original Image

Bayes Formulas

Posterior Probability

Degradation Process

\[ P(g | f) \]

Degraded Image

\[ P(f | g) = \frac{P(g | f)P(f)}{P(g)} \]

\( E \): Set of all the nearest neighbour pairs of pixels

\[ \hat{f}_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_i P(f_1, f_2, \ldots, f_{|V|} | g_1, g_2, \ldots, g_{|V|}) df_1 df_2 \cdots df_{|V|} \]

Image processing is reduced to calculations of averages, variances and co-variances in the posterior probability.
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Bayesian Image Analysis by Gaussian Graphical Model

Prior Probability

\[ P(f_1, f_2, \ldots, f_{|V|}) \propto \exp \left( -\frac{1}{2} \alpha \sum_{\{i, j\} \in E} (f_i - f_j)^2 \right) \]

\[ f_i \in (-\infty, +\infty) \]

Patterns are generated by MCMC.

\[ \alpha = 0.0001 \quad \alpha = 0.0005 \quad \alpha = 0.0030 \]

V: Set of all the pixels

E: Set of all the nearest-neighbour pairs of pixels
Bayesian Image Analysis by Gaussian Graphical Model

Degradation Process is assumed to be the additive white Gaussian noise.

\[ P(g_1, g_2, \ldots, g_N \mid f_1, f_2, \ldots, f_N) = \prod_{i \in V} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{1}{2\sigma^2}(g_i - f_i)^2 \right) \]

Degraded image is obtained by adding a white Gaussian noise to the original image.

\[ g_i \sim N(0, \sigma^2) \]

\[
\begin{array}{ccc}
\text{Original Image} & \text{Gaussian Noise} & \text{Degraded Image} \\
\end{array}
\]

Histogram of Gaussian Random Numbers
Bayesian Image Analysis by Gaussian Graphical Model

Posterior Probability

\[ f_i, g_i \in (-\infty, +\infty) \]

Average of the posterior probability can be calculated by using the multi-dimensional Gauss integral Formula

\[ \hat{f} = \int f P(f | g) df = (I + \alpha \sigma^2 C)^{-1} g \]

Multi-Dimensional Gaussian Integral Formula

\[ P(f | g) = \frac{P(g | f)P(f)}{P(g)} \]

\[ \propto \exp \left( -\frac{1}{2\sigma^2} \sum_i (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{\{i,j\}} (f_i - f_j)^2 \right) \]

\[ V: \text{Set of all the pixels} \]

\[ E: \text{Set of all the nearest-neighbour pairs of pixels} \]

\[ \langle i | C | j \rangle = \begin{cases} 4 & (i = j) \\ -1 & (ij \in E) \\ 0 & \text{(otherwise)} \end{cases} \]

\[ |V|x|V| \text{ matrix} \]
Bayesian Image Analysis
by Gaussian Graphical Model

Iteration Procedure of EM algorithm in Gaussian Graphical Model

$$\begin{align*}
(\hat{\alpha}, \hat{\sigma}) &= \arg\max_{(\alpha, \sigma)} P(g \mid \alpha, \sigma) \\
(\alpha(t + 1), \sigma(t + 1)) &\leftarrow Q(\alpha(t), \sigma(t), g) \\
\hat{f}(t) &= (I + \alpha(t)\sigma(t)^2 C)^{-1} g
\end{align*}$$
Image Restoration by Gaussian Graphical Model and Conventional Filters

<table>
<thead>
<tr>
<th>Method</th>
<th>Filter Size</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistical Method</td>
<td>(3x3)</td>
<td>315</td>
</tr>
<tr>
<td></td>
<td>(5x5)</td>
<td>388</td>
</tr>
<tr>
<td>Lowpass Filter</td>
<td>(3x3)</td>
<td>486</td>
</tr>
<tr>
<td></td>
<td>(5x5)</td>
<td>445</td>
</tr>
<tr>
<td>Median Filter</td>
<td>(3x3)</td>
<td>388</td>
</tr>
<tr>
<td></td>
<td>(5x5)</td>
<td>445</td>
</tr>
</tbody>
</table>

\[
\text{MSE} = \frac{1}{|V|} \sum_{i \in V} (f_i - \hat{f}_i)^2 \\
V: \text{Set of all the pixels}
\]
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Graphical Representation for Tractable Models

Tractable Model

\[ \sum_{A=0,1} \sum_{B=0,1} \sum_{C=0,1} f(A,D)g(B,D)h(C,D) \]

It is possible to calculate each summation independently.

Intractable Model

\[ \sum_{A=0,1} \sum_{B=0,1} \sum_{C=0,1} f(A,B)g(B,C)h(C,A) \]

It is hard to calculate each summation independently.
Loopy Belief Propagation for Graphical Model in Image Processing

Graphical model for image processing is represented in terms of the square lattice. Square lattice includes a lot of cycles. Belief propagation are applied to the calculation of statistical quantities as an approximate algorithm.

Loopy Belief Propagation

Every graph consisting of a pixel and its four neighbouring pixels can be regarded as a tree graph.
Loopy Belief Propagation in Image Processing

Each message passing rule includes 3 incoming messages and 1 outgoing message.

We have four kinds of message passing rules for each pixel.
EM algorithm by means of Belief Propagation

EM Algorithm for Hyperparameter Estimation

\[(\alpha(t + 1), \sigma(t + 1)) \leftarrow Q(\alpha(t), \sigma(t), g)\]

Update Rule of Loopy Belief Propagation
Probabilistic Image Processing by EM Algorithm and Loopy BP for Gaussian Graphical Model

\[
(\alpha(t + 1), \sigma(t + 1)) \leftarrow Q(\alpha(t), \sigma(t), g).
\]

\[
\hat{\sigma}_{\text{Exact}} = 37.624 \\
\hat{\alpha}_{\text{Exact}} = 0.000713
\]

\[
\hat{\sigma}_{\text{LBP}} = 36.335 \\
\hat{\alpha}_{\text{LBP}} = 0.000600
\]

MSE: 315

MSE: 327
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Digital Images Inpainting based on MRF

Markov Random Fields

M. Yasuda, J. Ohkubo and K. Tanaka:
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Summary

- Formulation of probabilistic model for image processing by means of conventional statistical schemes has been summarized.
- Probabilistic image processing by using Gaussian graphical model has been shown as the most basic example.
- It has been explained how to construct a belief propagation algorithm for image processing.
References


SMAPIP Project
Statistical Mechanical Approach to Probabilistic Information Processing

Member:
K. Tanaka, Y. Kabashima,
H. Nishimori, T. Tanaka,
M. Okada, O. Watanabe,
N. Murata, ......

Period: 2002 – 2005
Head Investigator: Kazuyuki Tanaka

MEXT Grant-in Aid for Scientific Research on Priority Areas
DEX-SMI Project
Deepening and Expansion of Statistical Mechanical Informatics

MEXT Grant-in Aid for Scientific Research on Priority Areas
Period: 2006 – 2009
Head Investigator: Prof. Yoshiyuki Kabashima (Tokyo Institute of Technology)

Member: Y. Kabashima, K. Tanaka, H. Nishimori, T. Tanaka, M. Okada, S. Ishii, M. Hayashi,…

http://dex-smi.sp.dis.titech.ac.jp/DEX-SMI/
CERIES Project
Center of Education and Research for Information Electronics Systems
GCOE Program

Period: 2007 – 2011
Head Investigator: Prof. Fumiyuki Adachi (Tohoku University)

http://www.ecei.tohoku.ac.jp/gcoe/