First-Order Phase Transition and Bayesian Image Processing by Loopy Belief Propagation

Kazuyuki TANAKA[1*)] and D. M. TITTERINGTON[2**)]

[1]*Graduate School of Information Sciences, Tohoku University, Aramaki-aza-aoba 09, Aoba-ku, Sendai 980-8579, Japan*
[2]*Department of Statistics, University of Glasgow, Glasgow G12 8QQ, UK*

The framework is presented of Bayesian image restoration for multi-valued images based on the Q-state Potts model. Hyperparameters in the probabilistic model are determined so as to maximize the marginal likelihood. A practical algorithm is described for multi-valued image restoration based on loopy belief propagation. We conclude that the maximization of marginal likelihood can provide good results even if the *a priori* probabilistic model exhibits first-order phase transition.

§1. Introduction

In Bayesian image restoration, we have to determine certain hyperparameters, and in practice it is common for the hyperparameters to be determined so as to maximize a marginal likelihood[1)]. In some probabilistic models treated in statistical mechanics, some types of phase transition occur at a critical point. First-order and second-order phase transitions are familiar types of phase transition. In the case of first-order phase transition, the first derivative of the free energy is discontinuous at the critical point; if the probabilistic model exhibits first-order phase transition, the free energy is not differentiable at a critical value of the hyperparameter. The marginal likelihood can be expressed in terms of the free energies of the *a priori* and *a posteriori* probability distributions. This means that the marginal likelihood is not differentiable at a critical value of the hyperparameter present in the *a priori* probability distribution. In the context of the expectation-maximization algorithm, which is a powerful method for maximizing the marginal likelihood, the marginal likelihood is differentiable at any value of the hyperparameter.

In the present paper, we investigate Bayesian image restoration with hyperparameters estimated by maximizing the marginal likelihood when we adopt a Q-state Potts model[2)] as the *a priori* probability distribution. The free energy of the Q-state Potts model for $Q \geq 3$ is not differentiable at a certain value of the hyperparameter when we analyze the model by means of advanced mean-field approximations. The maximization of the marginal likelihood is achieved by using loopy belief propagation[1),4),6)].

*) URL: http://www.statp.is.tohoku.ac.jp/˜kazu/, E-mail: kazu@statp.is.tohoku.ac.jp
**) URL: http://www.stats.gla.ac.uk/˜mike/, E-mail: mike@stats.gla.ac.uk

§2. Bayesian Image Analysis based on the Q-state Potts Model

We consider an image on a square lattice $\Omega\equiv\{i\}$ such that each pixel takes one of the gray-levels $\boldsymbol{Q}=\{0,1,2,\cdots,Q-1\}$, with 0 and $Q-1$ corresponding to black and white, respectively. The intensities at pixel i in the original image and the degraded image are regarded as random variables denoted by F_i and G_i, respectively, and the random fields of intensities in the original image and the degraded image are represented by $\boldsymbol{F}\equiv\{F_i|i\in\Omega\}$ and $\boldsymbol{G}\equiv\{G_i|i\in\Omega\}$, respectively. The actual original image and degraded image are denoted by $\boldsymbol{f}=\{f_i\}$ and $\boldsymbol{g}=\{f_i\}$, respectively.

In the present paper, it is assumed that the degraded image $\boldsymbol{g}$ is generated from the original image $\boldsymbol{f}$ by changing the intensity of each pixel to another intensity with the same probability p, independently of the other pixels; the probability that a pixel's intensity is unchanged is therefore $1-(Q-1)p$, which constrains p to lie in the range $0\le p\le 1/Q$. The conditional probability that the degraded image is $\boldsymbol{g}$ when the original image is $\boldsymbol{f}$ is denoted by $\Pr\{\boldsymbol{G}=\boldsymbol{g}\big|\boldsymbol{F}=\boldsymbol{f},p\}$. Moreover, the *a priori* probability distribution $\Pr\{\boldsymbol{F}=\boldsymbol{f}|\alpha\}$ that the original image is $\boldsymbol{f}$ is assumed to be a Q-state Potts model with the interaction parameter α between any nearest-neighbor pair of pixels. By substituting the *a priori* probability distribution and the conditional probability distribution for the degradation process into the Bayes formula, we obtain the *a posteriori* probability distribution $\Pr\{\boldsymbol{F}=\boldsymbol{f}|\boldsymbol{G}=\boldsymbol{g},\alpha,p\}$.

In the maximum marginal likelihood estimation approach, the hyperparameters α and p are determined so as to maximize the marginal likelihood $\Pr\{\boldsymbol{G}=\boldsymbol{g}|\alpha,p\}\equiv\sum_{\boldsymbol{z}}\Pr\{\boldsymbol{G}=\boldsymbol{g}|\boldsymbol{F}=\boldsymbol{z},p\}\Pr\{\boldsymbol{F}=\boldsymbol{z}|\alpha\}$. We denote the maximizers of the marginal likelihood $\Pr\{\boldsymbol{G}=\boldsymbol{g}|\alpha,p\}$ by $\widehat{\alpha}$ and $\widehat{p}$. The marginal likelihood $\Pr\{\boldsymbol{G}=\boldsymbol{g}|\alpha,p\}$ is expressed in terms of the free energies of the *a priori* and the *a posteriori* probability distributions. Given the estimates $\widehat{\alpha}$ and $\widehat{p}$, the restored image $\widehat{\boldsymbol{f}}=\{\widehat{f}_i|i\in\Omega\}$ is determined so as to maximize the posterior marginal probability $\Pr\{F_i=f_i|\boldsymbol{G}=\boldsymbol{g},\widehat{\alpha},\widehat{p}\}\equiv\sum_{\boldsymbol{z}}\delta_{f_i,z_i}\Pr\{\boldsymbol{F}=\boldsymbol{z}|\boldsymbol{G}=\boldsymbol{g},\widehat{\alpha},\widehat{p}\}$.[*] This way of producing a restored image is called maximum posterior marginal estimation.

§3. Numerical Experiments

In this section, we report some numerical experiments. The optimal set of values of the hyperparameters, $(\widehat{\alpha},\widehat{p})$, are determined by means of maximum marginal likelihood estimation as described in Section 2 and loopy belief propagation[3),4),6)].

It is well known that the first derivative of the free energy of the Q-state Potts model for $Q\ge 3$ is discontinuous at a value of α when we analyse it by using advanced mean-field approximations[2)]. In the calculation of loopy belief propagation, the derivative with respect to α of the free energy for $Q=4$ is discontinuous at $\alpha\simeq 1.0050$ though the free energy is continuous at any value of α. The Potts model exhibits first-order phase transition when we calculate the approximate free energy by using loopy belief propagation.

To evaluate restoration performance quantitatively, fifteen original images $\boldsymbol{f}$ are

[*] $\delta_{a,b}$ is the Kronecker delta.

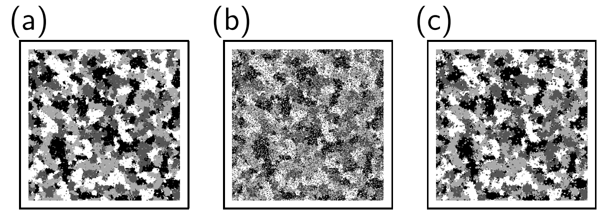

Fig. 1. Image restoration based on the Q-state Potts model ($Q = 4$). The original image $\boldsymbol{f}$ is generated by the *a priori* probability distribution. (a) Original image $\boldsymbol{f}$ ($\alpha = 1.25$). (b) Degraded image $\boldsymbol{g}$ ($(Q-1)p = 0.3$). (c) Restored image $\widehat{\boldsymbol{f}}$ obtained by loopy belief propagation ($(Q-1)\widehat{p} = 0.2246$, $\widehat{\alpha} = 1.0050$, $d_{\mathrm{HD}}(\boldsymbol{f}, \widehat{\boldsymbol{f}}) = 0.1557$).

simulated from the *a priori* probability distribution for the Q-state Potts model. We produce a degraded image $\boldsymbol{g}$ from each original image $\boldsymbol{f}$ by means of the degradation process for $(Q-1)p = 0.3$. By applying the iterative algorithm of loopy belief propagation to each degraded image $\boldsymbol{g}$, we obtain estimates of the hyperparameters $\widehat{p}$ and $\widehat{\alpha}$ and the restored image $\widehat{\boldsymbol{f}}$. For $Q = 4$, one of the numerical experiments is shown in Figure 1. In order to show how the maximization of the marginal likelihood is achieved, we display, in Figure 2, $\ln \Pr\{\boldsymbol{G} = \boldsymbol{g} | \alpha, p\}$ as a function separately of p and α corresponding to the image restorations shown in Figure 1. From the fifteen degraded images $\boldsymbol{g}$ and the corresponding restored images $\widehat{\boldsymbol{f}}$, we calculate 95% confidence intervals for the hyperparameters, $\widehat{p}$ and $\widehat{\alpha}$, and the values of the Hamming distance $d_{\mathrm{HD}}(\boldsymbol{f}, \widehat{\boldsymbol{f}}) \equiv \frac{1}{|\Omega|} \sum_{i\in\Omega} \big(1 - \delta_{f_i, \widehat{f}_i}\big)$ and the mean square error $d_{\mathrm{MSE}}(\boldsymbol{f}, \widehat{\boldsymbol{f}}) \equiv \frac{1}{|\Omega|} \sum_{i\in\Omega} \big(f_i - \widehat{f}_i\big)^2$. These confidence intervals are given in Table I. These results show that combining loopy belief propagation with maximum

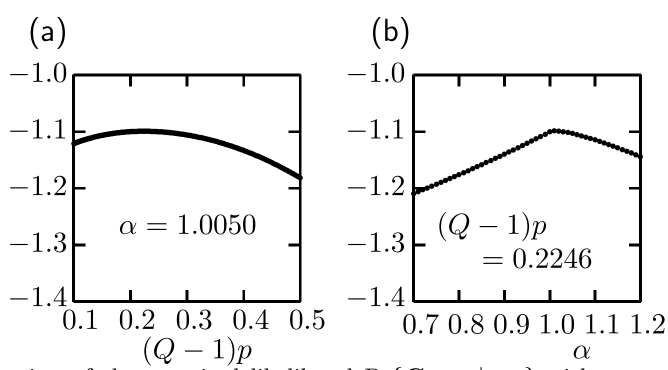

Fig. 2. Maximization of the marginal likelihood $\Pr\{\boldsymbol{G} = \boldsymbol{g} | \alpha, p\}$ with respect to p and α, corresponding to the image restoration shown in Figure 1(c). (a) p-dependence of the logarithm of the marginal likelihood $\frac{1}{|\Omega|} \ln\big(\Pr\{\boldsymbol{G} = \boldsymbol{g} | \alpha, p\}\big)$ with α set to 1.0050. (b) α-dependence of the logarithm of marginal likelihood $\frac{1}{|\Omega|} \ln\big(\Pr\{\boldsymbol{G} = \boldsymbol{g} | \alpha, p\}\big)$ with $(Q-1)p$ set to 0.2246.

marginal likelihood estimation gives us good results for hyperparameter estimation for the Q-state Potts model. Clearly the true values of the hyperparameters, α and p, lie outside the 95% confidence intervals. However, the results obtained by the loopy belief propagation represent systematic improvements over those obtained by the mean-field approximation[5),6)]. We have already reported elsewhere[5),6)] some

Table I. Approximate 95% confidence intervals for the hyperparameters, p and α, and the values of $d_{\mathrm{HD}}(\boldsymbol{f},\widehat{\boldsymbol{f}})$ and $d_{\mathrm{MSE}}(\boldsymbol{f},\widehat{\boldsymbol{f}})$ obtained for some degraded images $\boldsymbol{g}$, which are produced for $Q=4$ and $(Q-1)p=0.3$ from fifteen original images $\boldsymbol{f}$. The fifteen original images $\boldsymbol{f}$ are generated by Monte Carlo simulation from the *a priori* probability distributions for the Q-state Potts models. The hyperparameters are estimated by applying loopy belief propagation to maximum marginal likelihood estimation.

α	1.25	$\widehat{\alpha}$	[1.0050±0.0000]
$(Q-1)p$	0.30	$(Q-1)\widehat{p}$	[0.2246±0.0009]
$d_{\mathrm{HD}}(\boldsymbol{f},\boldsymbol{g})$	[0.2980±0.0005]	$d_{\mathrm{HD}}(\boldsymbol{f},\widehat{\boldsymbol{f}})$	[0.1580±0.0017]
$d_{\mathrm{MSE}}(\boldsymbol{f},\boldsymbol{g})$	[0.9935±0.0090]	$d_{\mathrm{MSE}}(\boldsymbol{f},\widehat{\boldsymbol{f}})$	[0.5255±0.0073]

good results from numerical experiments for probabilistic image restoration based on artificial images.

§4. Concluding Remarks

We have investigated probabilistic image restoration when the Q-state Potts model is adopted as the *a priori* probability distribution. Our algorithm was constructed by using loopy belief propagation and the hyperparameters are determined so as to maximize the marginal likelihood for each degraded image. The Potts model with three or more possible states at each pixel exhibits first-order phase transition when we treat it by using loopy belief propagation. The marginal likelihood is expressed in terms of the free energies of the *a posteriori* probabilistic model and the *a priori* probabilistic model. In the case of first-order phase transition, the free energy is not differentiable at a certain value of the hyperparameter. We have investigated how the marginal likelihood is maximized when the *a priori* probabilistic model exhibits first-order phase transition. We conclude that the Q-state Potts model can provide good results even though it exhibits first-order phase transition.

Acknowledgements

The authors are grateful to Professor T. Horiguchi of the Graduate School of Information Sciences, Tohoku University for valuable discussion. This work was partly supported by the Grants-In-Aid (No.14084203) for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

1) K. Tanaka, J. Inoue and D. M. Titterington, J.Phys. A: Math. & Gen. 36 (2003), 11023.
2) F. Y. Wu, The Potts model, Reviews of Modern Physics 54 (1982), 235.
3) M. Opper and D. Saad (edited), *Advanced Mean Field Methods - Theory and Practice*, MIT Press (2001).
4) J. S. Yedidia, W. T. Freeman and Y. Weiss, Advances in Neural Information Processing Systems 13 (2001), 689.
5) K. Tanaka, Electronics and Communications in Japan Part 3 85 (2002), 50.
6) K. Tanaka, J. Inoue and D. M. Titterington, Neural Networks for Signal Processing XIII (2003), 329.