

Bayesian Network and Probabilistic Image Processing --- Statistical Aspect of Belief Propagation Method ---

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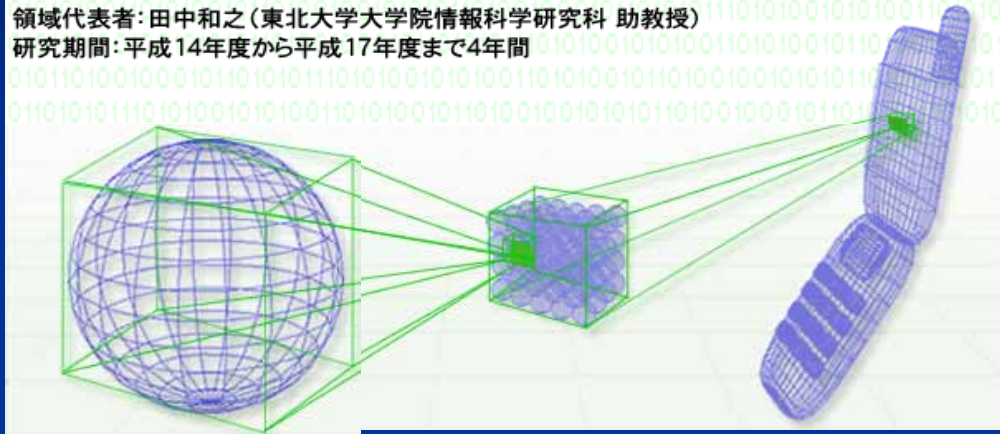
References

- **K. Tanaka: Statistical-mechanical approach to image processing (Topical Review), J. Phys. A, vol.35, pp.R81-R150 (2002).**
- **K. Tanaka, H. Shouno, M. Okada and D. M. Titterington: Accuracy of the Bethe approximation for hyperparameter estimation in probabilistic image processing, J. Phys. A, vol.37, pp.8675-8695 (2004).**

What is SMAPIP?

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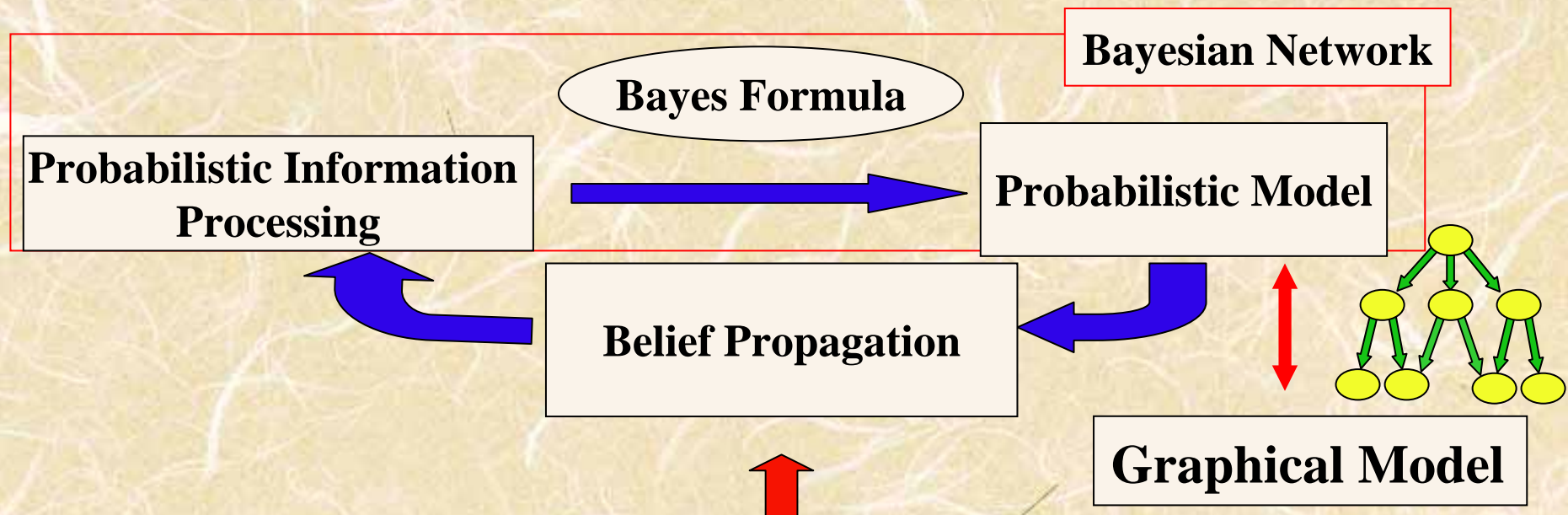
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Bayesian Network and Belief Propagation



J. Pearl: Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference (Morgan Kaufmann, 1988).
C. Berrou and A. Glavieux: Near optimum error correcting coding and decoding: Turbo-codes, IEEE Trans. Comm., 44 (1996).

Formulation of Belief Propagation

- **Link between belief propagation and statistical mechanics.**

Y. Kabashima and D. Saad, Belief propagation vs. TAP for decoding corrupted messages, *Europhys. Lett.* 44 (1998).

M. Opper and D. Saad (eds), *Advanced Mean Field Methods --- Theory and Practice* (MIT Press, 2001).

- **Generalized belief propagation**

J. S. Yedidia, W. T. Freeman and Y. Weiss: Constructing free-energy approximations and generalized belief propagation algorithms, *IEEE Transactions on Information Theory*, 51 (2005).

- **Information geometrical interpretation of belief propagation**

S. Ikeda, T. Tanaka and S. Amari: Stochastic reasoning, free energy, and information geometry, *Neural Computation*, 16 (2004).

Extension of Belief Propagation

- **Generalized Belief Propagation**

J. S. Yedidia, W. T. Freeman and Y. Weiss: Constructing free-energy approximations and generalized belief propagation algorithms, *IEEE Transactions on Information Theory*, 51 (2005).

- **Generalized belief propagation is equivalent to the cluster variation method in statistical mechanics**

R. Kikuchi: A theory of cooperative phenomena, *Phys. Rev.*, 81 (1951).

T. Morita: Cluster variation method of cooperative phenomena and its generalization I, *J. Phys. Soc. Jpn*, 12 (1957).

Application of Belief Propagation

• Image Processing

K. Tanaka: Statistical-mechanical approach to image processing (Topical Review), *J. Phys. A*, **35** (2002).

A. S. Willsky: Multiresolution Markov Models for Signal and Image Processing, *Proceedings of IEEE*, **90** (2002).

• Low Density Parity Check Codes

Y. Kabashima and D. Saad: Statistical mechanics of low-density parity-check codes (Topical Review), *J. Phys. A*, **37** (2004).

S. Ikeda, T. Tanaka and S. Amari: Information geometry of turbo and low-density parity-check codes, *IEEE Transactions on Information Theory*, **50** (2004).

• CDMA Multiuser Detection Algorithm

Y. Kabashima: A CDMA multiuser detection algorithm on the basis of belief propagation, *J. Phys. A*, **36** (2003).

T. Tanaka and M. Okada: Approximate Belief propagation, density evolution, and statistical neurodynamics for CDMA multiuser detection, *IEEE Transactions on Information Theory*, **51** (2005).

• Satisfiability Problem

O. C. Martin, R. Monasson, R. Zecchina: Statistical mechanics methods and phase transitions in optimization problems, *Theoretical Computer Science*, **265** (2001).

M. Mezard, G. Parisi, R. Zecchina: Analytic and algorithmic solution of random satisfiability problems, *Science*, **297** (2002).

Contents

1. Introduction
2. Belief Propagation
3. Bayesian Image Analysis and Gaussian Graphical Model
4. Concluding Remarks



Belief Propagation

- How should we treat the calculation of the summation over 2^N configuration?

$$\sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_N=0,1} W(x_1, x_2, \dots, x_N)$$

It is very hard to calculate exactly except some special cases.

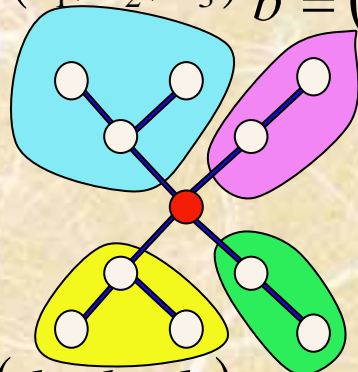
- Formulation for approximate algorithm
- Accuracy of the approximate algorithm

Tractable Model

● Probabilistic models with no loop are tractable.

$$\vec{a} = (a_1, a_2, a_3) \quad \vec{b} = (b_1, b_2)$$

Factorizable

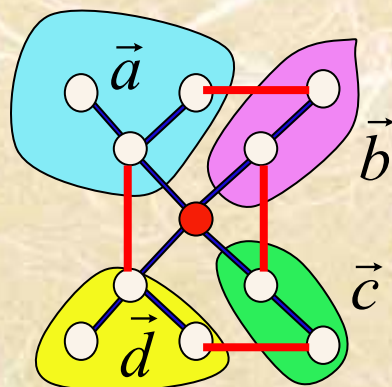


$$\vec{d} = (d_1, d_2, d_3) \quad \vec{c} = (c_1, c_2)$$

$$\sum_{\vec{a}} \sum_{\vec{b}} \sum_{\vec{c}} \sum_{\vec{d}} A(\vec{a}, x) B(\vec{b}, x) C(\vec{c}, x) D(\vec{d}, x)$$

$$= \left(\sum_{\vec{a}} A(\vec{a}, x) \right) \left(\sum_{\vec{b}} B(\vec{b}, x) \right) \left(\sum_{\vec{c}} C(\vec{c}, x) \right) \left(\sum_{\vec{d}} D(\vec{d}, x) \right)$$

● Probabilistic models with loop are not tractable.



Not Factorizable

$$\sum_{\vec{a}} \sum_{\vec{b}} \sum_{\vec{c}} \sum_{\vec{d}} W(\vec{a}, \vec{b}, \vec{c}, \vec{d}, x)$$

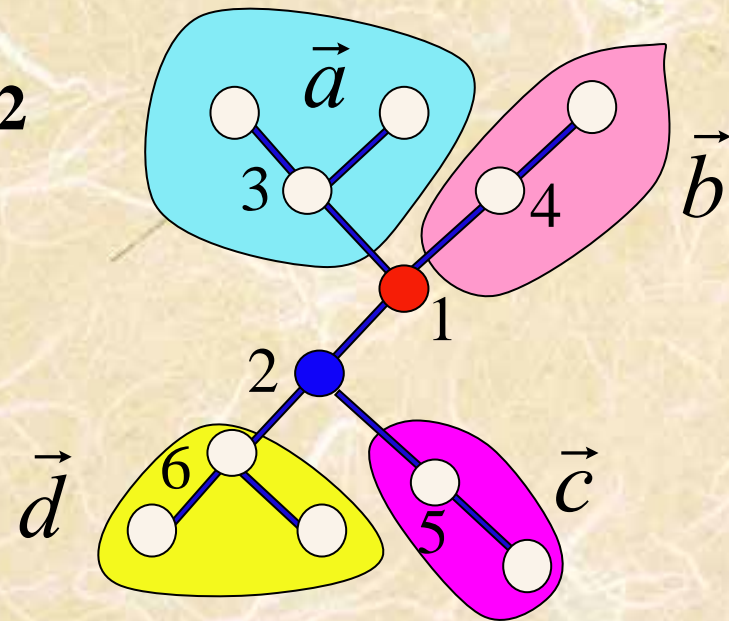
Probabilistic model on a graph with no loop

$$P(\vec{a}, \vec{b}, \vec{c}, \vec{d}, x, y)$$

$$\equiv W_A(\vec{a}, x)W_B(\vec{b}, x)W_{12}(x, y)W_C(\vec{c}, y)W_D(\vec{d}, y)$$

$$P_2(y) \equiv \sum_{\vec{a}} \sum_{\vec{b}} \sum_{\vec{c}} \sum_{\vec{d}} \sum_x P(\vec{a}, \vec{b}, \vec{c}, \vec{d}, x, y)$$

Marginal probability of the node 2

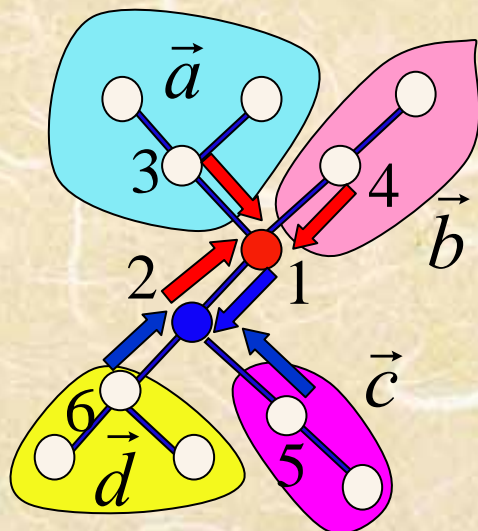


Probabilistic model on a graph with no loop

$$P_2(y) = M_{5 \rightarrow 2}(y) M_{6 \rightarrow 2}(y) M_{1 \rightarrow 2}(y)$$

$$M_{1 \rightarrow 2}(y) = \sum_x W_{12}(x, y) M_{3 \rightarrow 1}(x) M_{4 \rightarrow 1}(x)$$

$$M_{2 \rightarrow 1}(x) = \sum_y W_{12}(x, y) M_{5 \rightarrow 2}(y) M_{6 \rightarrow 2}(y)$$

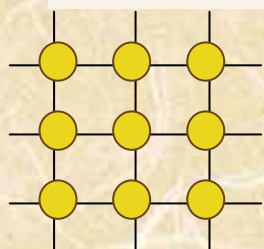


Marginal probability can be expressed in terms of the product of messages from all the neighbouring nodes of node 2.

Message from the node 1 to the node 2 can be expressed in terms of the product of message from all the neighbouring nodes of the node 1 except one from the node 2.

Probabilistic Model on a Graph with Loops

$$P(x_1, x_2, \dots, x_L) \equiv \frac{1}{Z} \prod_{ij \in N} W_{ij}(x_i, x_j)$$



Ω

$$Z \equiv \sum_{x_1} \sum_{x_2} \dots \sum_{x_L} \prod_{ij \in N} W_{ij}(x_i, x_j)$$

Marginal Probability

$$P_1(x_1) \equiv \sum_{x_2} \sum_{x_3} \sum_{x_4} \dots \sum_{x_L} P(x_1, x_2, x_3, x_4, \dots, x_L)$$

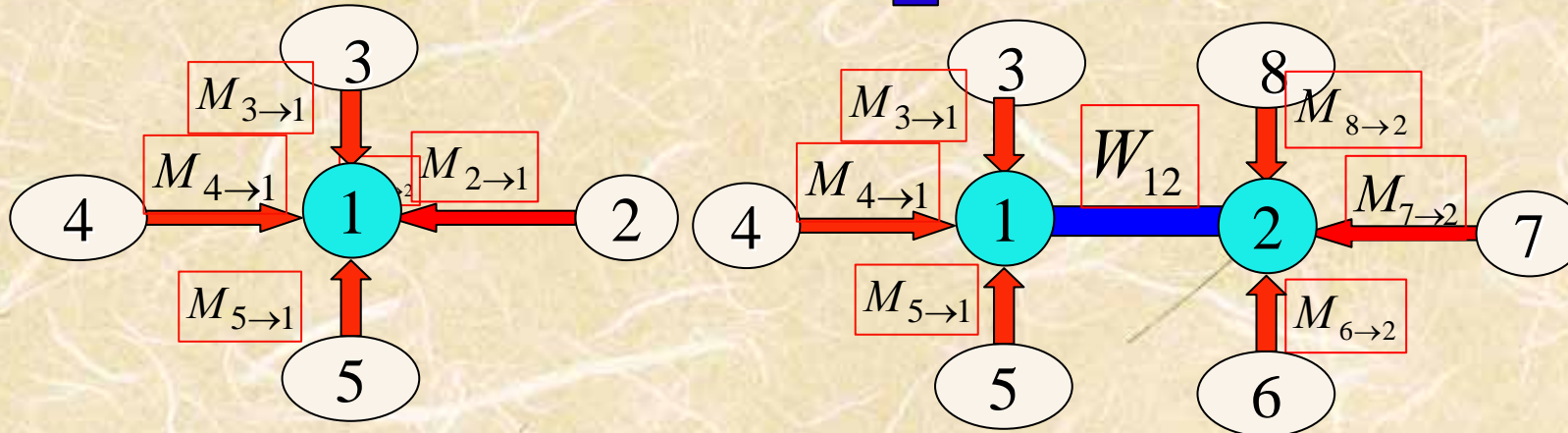
$$P_{12}(x_1, x_2) \equiv \sum_{x_3} \sum_{x_4} \dots \sum_{x_L} P(x_1, x_2, x_3, x_4, \dots, x_L)$$

Belief Propagation

$$P_1(\xi) = \sum_{\zeta} P_{12}(\xi, \zeta)$$

$$\propto \sum_{\zeta} W_{12}(\zeta, \xi) M_{3 \rightarrow 1}(\zeta) \times M_{4 \rightarrow 1}(\zeta) M_{5 \rightarrow 1}(\zeta)$$

Message Update Rule

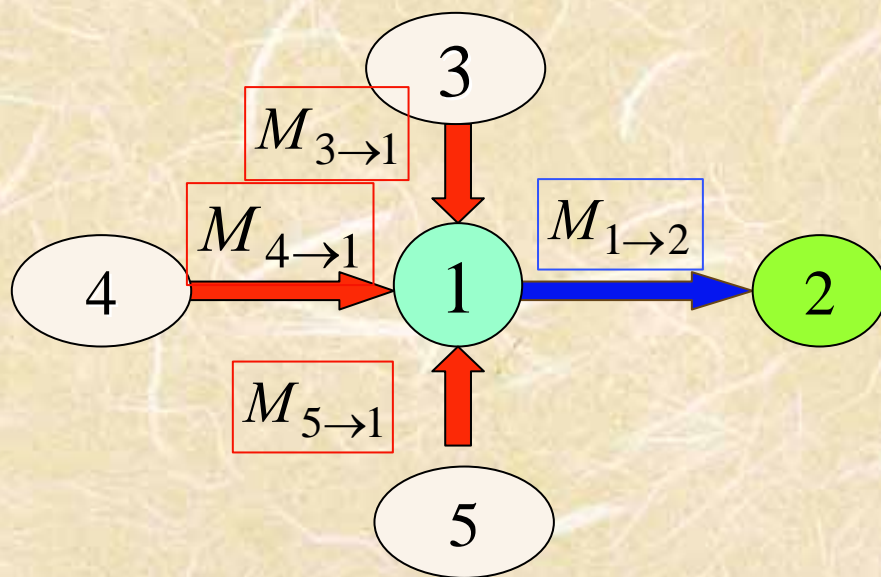


$$P_1(x_1) \cong \frac{1}{Z_1} M_{2 \rightarrow 1}(x_1) M_{3 \rightarrow 1}(x_1) \times M_{4 \rightarrow 1}(x_1) M_{5 \rightarrow 1}(x_1)$$

$$P_{12}(x_1, x_2) \cong \frac{1}{Z_{12}} M_{3 \rightarrow 1}(x_1) M_{4 \rightarrow 1}(x_1) M_{5 \rightarrow 1}(x_1) \times W_{12}(x_1, x_2) M_{6 \rightarrow 2}(x_2) M_{7 \rightarrow 2}(x_2) M_{8 \rightarrow 2}(x_2)$$

Message Passing Rule of Belief Propagation

$$M_{1 \rightarrow 2}(\xi) = \frac{\sum_{\zeta} W_{12}(\zeta, \xi) M_{3 \rightarrow 1}(\zeta) M_{4 \rightarrow 1}(\zeta) M_{5 \rightarrow 1}(\zeta)}{\sum_{\xi} \sum_{\zeta} W_{12}(\zeta, \xi) M_{3 \rightarrow 1}(\zeta) M_{4 \rightarrow 1}(\zeta) M_{5 \rightarrow 1}(\zeta)}$$



**Fixed Point Equations
for Message**

$$\vec{M} = \vec{\Phi}(\vec{M})$$

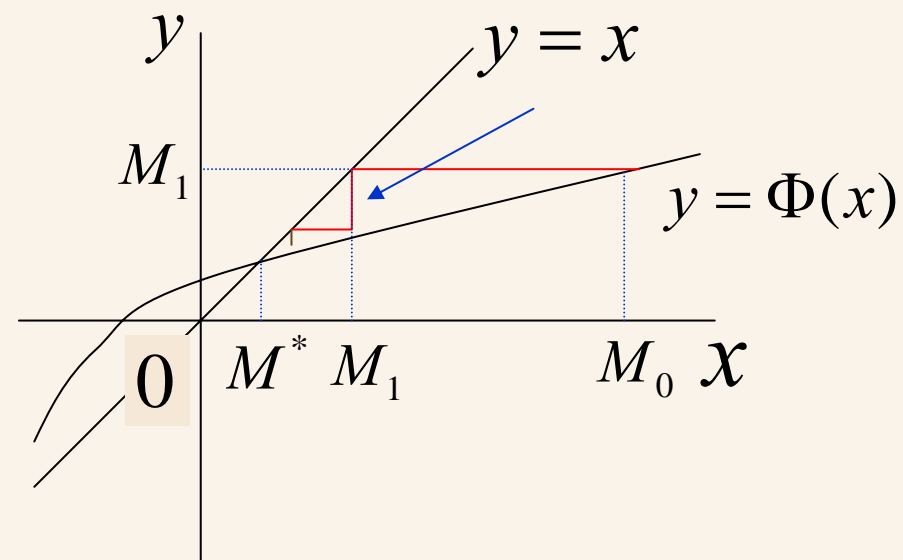
Fixed Point Equation and Iterative Method

Fixed Point Equation

$$\vec{M}^* = \vec{\Phi}(\vec{M}^*)$$

Iterative Method

$$\begin{aligned} \vec{M}_1 &\leftarrow \Phi(\vec{M}_0) \\ \vec{M}_2 &\leftarrow \Phi(\vec{M}_1) \\ \vec{M}_3 &\leftarrow \Phi(\vec{M}_2) \\ &\vdots \end{aligned}$$



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Bayesian Image Analysis



Original Image



Noise



Degraded Image



$$\underbrace{\Pr\{\text{Original Image}|\text{Degraded Image}\}}_{\text{A Posteriori Probability}} = \frac{\overbrace{\Pr\{\text{Degraded Image}|\text{Original Image}\}}^{\text{Degradation Process}} \overbrace{\Pr\{\text{Original Image}\}}^{\text{A Priori Probability}}}{\underbrace{\Pr\{\text{Degraded Image}\}}_{\text{Marginal Likelihood}}}$$

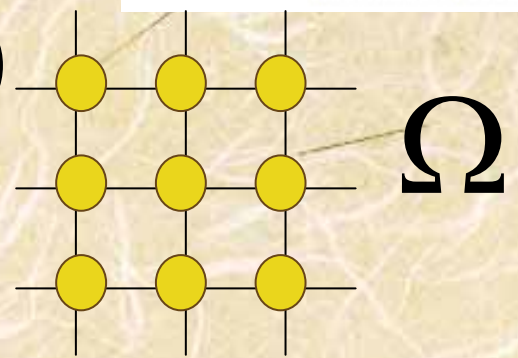
Bayesian Image Analysis

Degradation Process

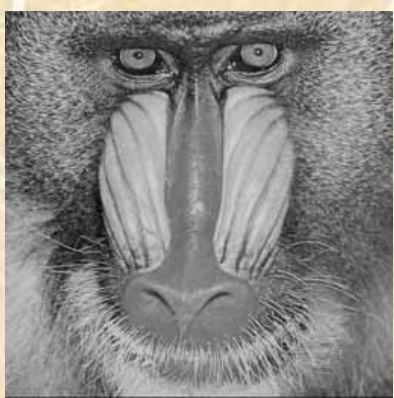
$$g_i = f_i + n_i$$

$$n_i \sim N(0, \sigma^2)$$

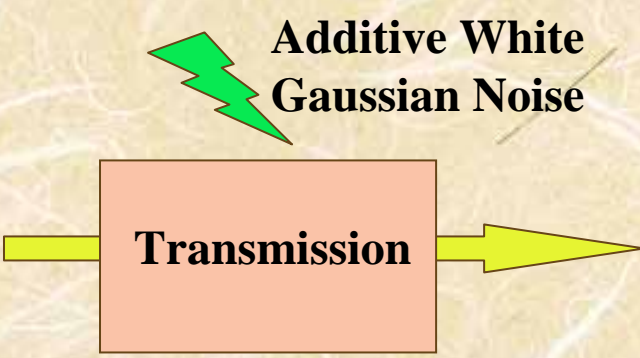
$$f_i, g_i \in (-\infty, +\infty)$$



$$P(\vec{g} | \vec{f}, \sigma) = \prod_{i \in \Omega} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} (f_i - g_i)^2\right)$$



Original Image



Degraded Image

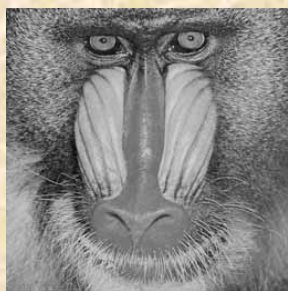
Bayesian Image Analysis

A Priori Probability

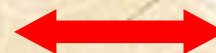
$$f_i \in (-\infty, +\infty)$$

$$P(\vec{f}|\alpha) = \frac{1}{Z_{\text{PR}}(\alpha)} \prod_{ij \in N} \exp\left(-\frac{1}{2} \alpha (f_i - f_j)^2\right)$$

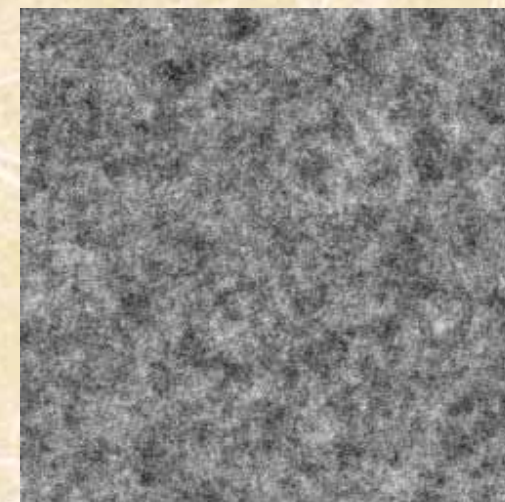
Generate



Standard Images



Similar?



Bayesian Image Analysis

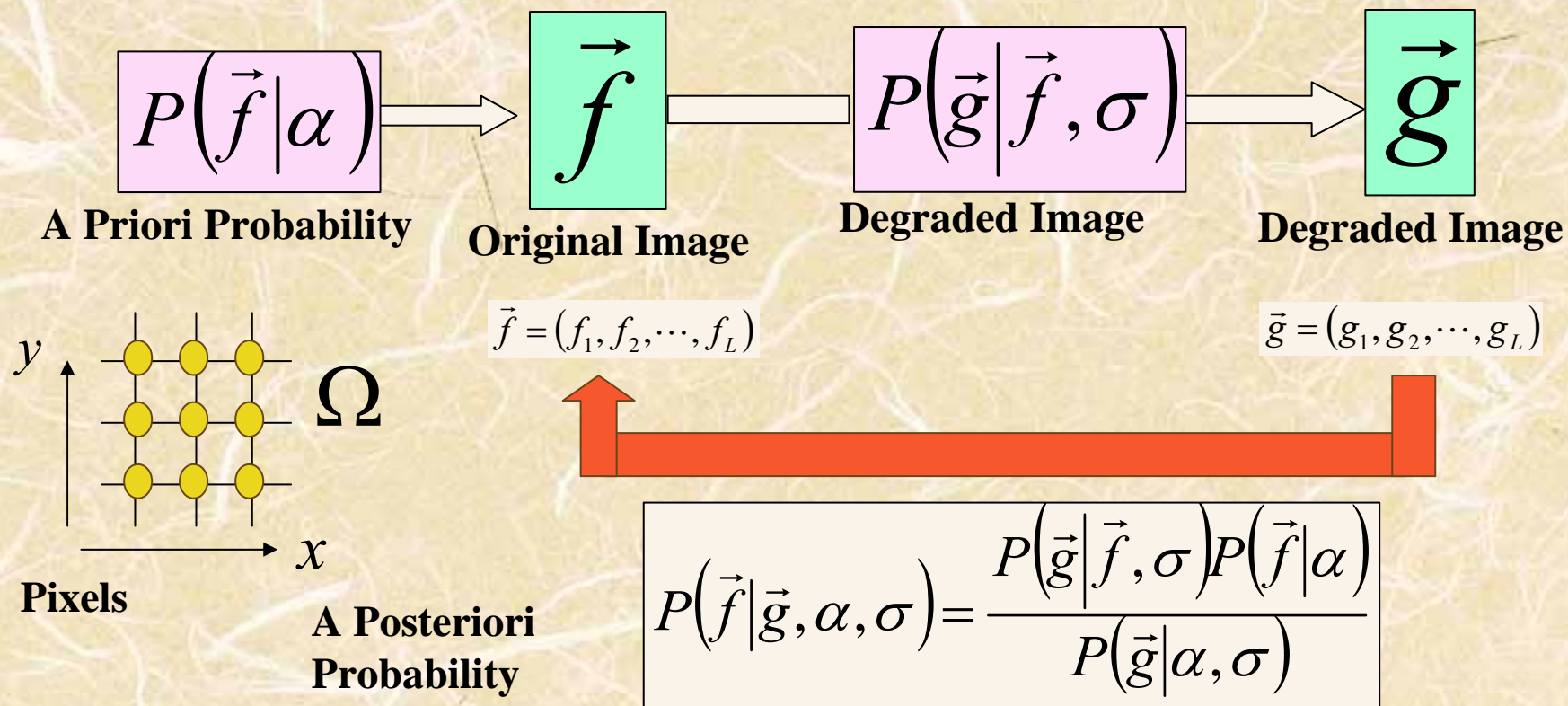
A Posteriori Probability

$$f_i, g_j \in (-\infty, +\infty)$$

$$\begin{aligned} P(\vec{f} | \vec{g}, \alpha, \sigma) &= \frac{P(\vec{g} | \vec{f}, \sigma) P(\vec{f} | \alpha)}{P(\vec{g} | \alpha, \sigma)} \\ &= \frac{1}{Z_{\text{POS}}(\vec{g}, \alpha, \sigma)} \\ &\quad \times \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in \Omega} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{ij \in N} (f_i - f_j)^2\right) \end{aligned}$$

Gaussian Graphical Model

Bayesian Image Analysis



$$\hat{f}_i = \int f_i P(\vec{f}|\vec{g}, \alpha, \sigma) d\vec{f} = \int_{-\infty}^{+\infty} f_i P(f_i|\vec{g}, \alpha, \sigma) df_i$$

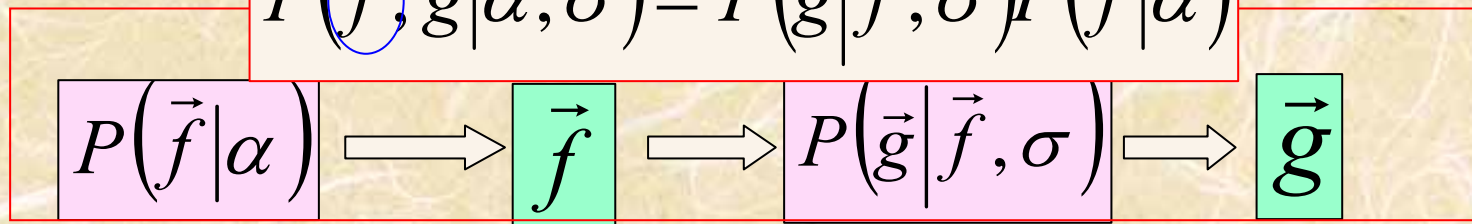
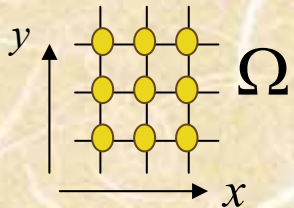
Hyperparameter Determination by Maximization of Marginal Likelihood

$$(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(\vec{g} | \alpha, \sigma)$$

$$\hat{f}_i = \int f_i P(\vec{f} | \vec{g}, \hat{\alpha}, \hat{\sigma}) d\vec{f}$$

$$P(\vec{g} | \alpha, \sigma) = \int P(\vec{f}, \vec{g} | \alpha, \sigma) d\vec{f} = \int P(\vec{g} | \vec{f}, \sigma) P(\vec{f} | \alpha) d\vec{f}$$

$$P(\vec{f}, \vec{g} | \alpha, \sigma) = P(\vec{g} | \vec{f}, \sigma) P(\vec{f} | \alpha)$$



Marginalization

Original Image
 $\vec{f} = (f_1, f_2, \dots, f_L)$

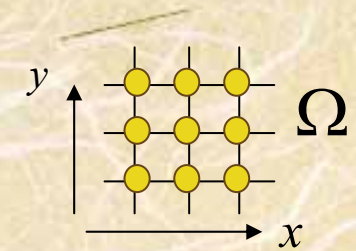
Marginal Likelihood
 $P(\vec{g} | \alpha, \sigma) \Rightarrow \vec{g}$

Degraded Image
 $\vec{g} = (g_1, g_2, \dots, g_L)$

Maximization of Marginal Likelihood by EM (Expectation Maximization) Algorithm

**Marginal
Likelihood**

$$P(\vec{g}|\alpha, \sigma) = \int P(\vec{g}|\vec{f}, \sigma) P(\vec{f}|\alpha) d\vec{f}$$



Q-Function

$$Q(\alpha', \sigma' | \alpha, \sigma, \vec{g}) = \int P(\vec{f} | \vec{g}, \alpha, \sigma) \ln P(\vec{f}, \vec{g} | \alpha', \sigma') d\vec{f}$$

$$\frac{\partial}{\partial \sigma} P(\vec{g} | \alpha, \sigma) = 0, \quad \frac{\partial}{\partial \alpha} P(\vec{g} | \alpha, \sigma) = 0$$

Incomplete Data

$$\vec{g} = (g_1, g_2, \dots, g_L)$$



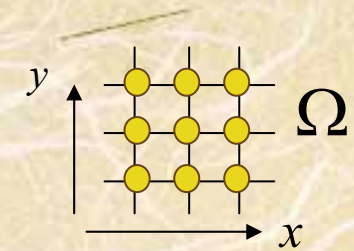
Equivalent

$$\left[\frac{\partial}{\partial \alpha'} Q(\alpha', \sigma' | \alpha, \sigma, \vec{g}) \right]_{\alpha'=\alpha, \sigma'=\sigma} = 0, \quad \left[\frac{\partial}{\partial \sigma'} Q(\alpha', \sigma' | \alpha, \sigma, \vec{g}) \right]_{\alpha'=\alpha, \sigma'=\sigma} = 0$$

Maximization of Marginal Likelihood by EM (Expectation Maximization) Algorithm

Marginal Likelihood

$$P(\vec{g}|\alpha, \sigma) = \int P(\vec{g}|\vec{f}, \sigma) P(\vec{f}|\alpha) d\vec{f}$$



Q-Function

$$Q(\alpha', \sigma'|\alpha, \sigma, \vec{g}) = \int P(\vec{f}|\vec{g}, \alpha, \sigma) \ln P(\vec{f}, \vec{g}|\alpha', \sigma') d\vec{f}$$

EM Algorithm

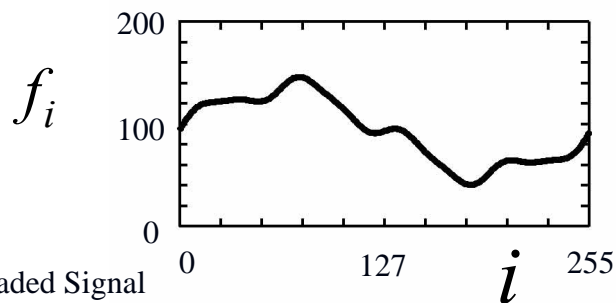
Iterate the following EM-steps until convergence:

E - Step : $Q(\alpha', \sigma'|\alpha(t), \sigma(t)) \leftarrow \int P(\vec{f}|\vec{g}, \alpha(t), \sigma(t)) \ln P(\vec{f}, \vec{g}|\alpha', \sigma') d\vec{f}$.

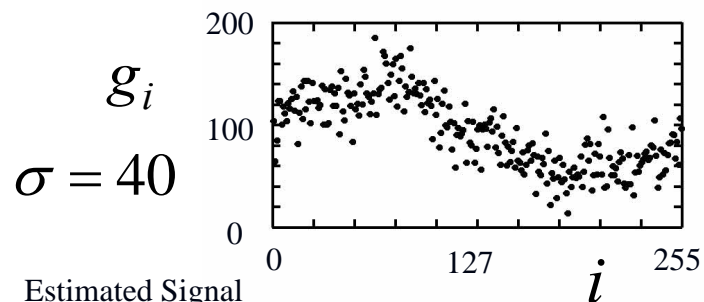
M - Step : $(\alpha(t+1), \sigma(t+1)) \leftarrow \arg \max_{(\alpha', \sigma')} Q(\alpha', \sigma'|\alpha(t), \sigma(t))$.

A. P. Dempster, N. M. Laird and D. B. Rubin, "Maximum likelihood from incomplete data via the EM algorithm," J. Roy. Stat. Soc. B, **39** (1977).

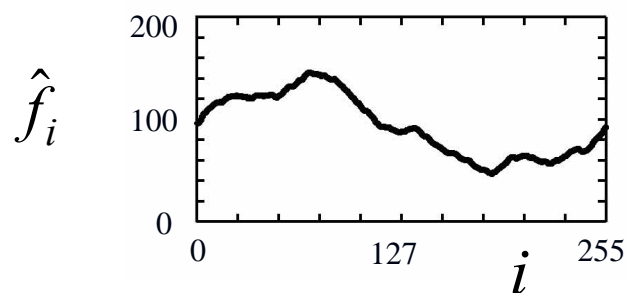
Original Signal



Degraded Signal



Estimated Signal



EM Algorithm

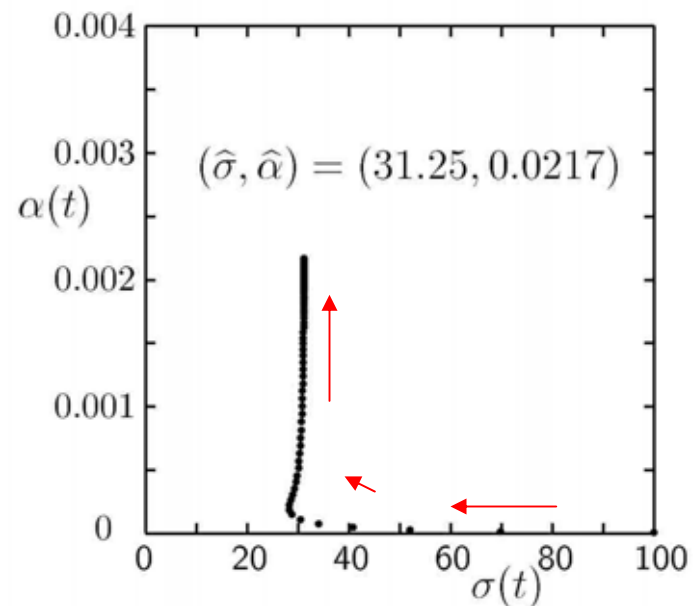
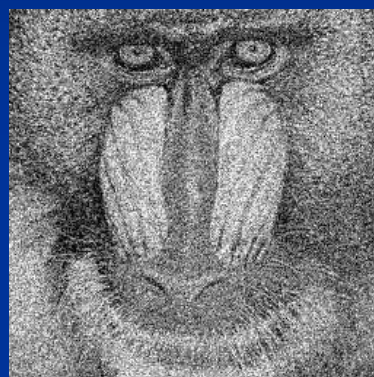


Image Restoration by Gaussian Graphical Model

Original Image



Degraded Image

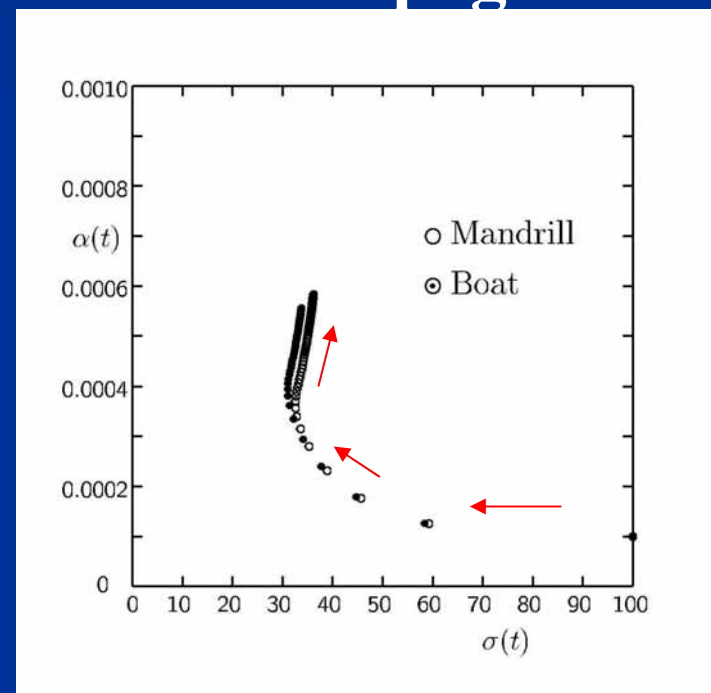


MSE: 1512



MSE: 1529

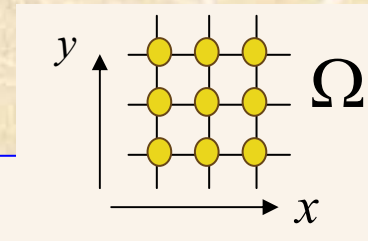
EM Algorithm with Belief Propagation



Approximate EM Algorithm of Gaussian Graphical Model by using Belief Propagation

$$P(\vec{f}, \vec{g} | \alpha, \sigma) \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in \Omega} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{ij \in N} (f_i - f_j)^2\right)$$

↓ Belief Propagation



EM Algorithm

Iterate the following EM-steps until convergence:

E - Step : $Q(\alpha', \sigma' | \alpha(t), \sigma(t)) \leftarrow \int P(\vec{f} | \vec{g}, \alpha(t), \sigma(t)) \ln P(\vec{f}, \vec{g} | \alpha', \sigma') d\vec{f}.$

M - Step : $(\alpha(t+1), \sigma(t+1)) \leftarrow \arg \max_{(\alpha', \sigma')} Q(\alpha', \sigma' | \alpha(t), \sigma(t)).$

Exact Results of Gaussian Graphical Model

$$P(\vec{f}, \vec{g} | \alpha, \sigma) \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i \in \Omega} (f_i - g_i)^2 - \frac{1}{2} \alpha \sum_{ij \in N} (f_i - f_j)^2\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2} \|\vec{f} - \vec{g}\|^2 - \frac{1}{2} \alpha \vec{f} \mathbf{C} \vec{f}^T\right)$$



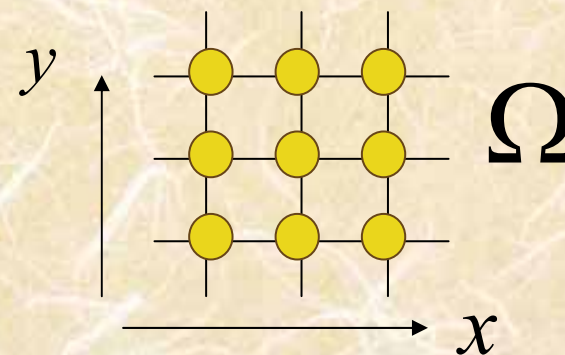
Multi-dimensional Gauss integral formula

$$P(\vec{g} | \alpha, \sigma) = \sqrt{\frac{\det(\alpha \mathbf{C})}{(2\pi)^{|\Omega|} \det(\mathbf{I} + \alpha \sigma^2 \mathbf{C})}} \exp\left(-\frac{1}{2} \alpha \vec{g} \frac{\mathbf{C}}{\mathbf{I} + \alpha \sigma^2 \mathbf{C}} \vec{g}^T\right)$$

We can construct an exact EM algorithm.

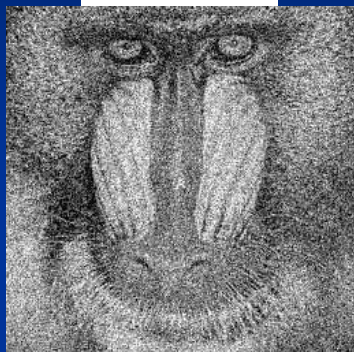
$$(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(\vec{g} | \alpha, \sigma)$$

$$\hat{\vec{f}} = (\mathbf{I} + \alpha \sigma^2 \mathbf{C})^{-1} \vec{g}$$



Comparison of Belief Propagation with Exact Results in Gaussian Graphical Model

$\sigma = 40$



$$(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(\vec{g} | \alpha, \sigma)$$

	MSE	$\hat{\alpha}$	$\hat{\sigma}$	$\ln P(g \hat{\alpha}, \hat{\sigma})$
Belief Propagation	327	0.000611	36.302	-5.19201
Exact	315	0.000759	37.919	-5.21444

$\sigma = 40$



	MSE	$\hat{\alpha}$	$\hat{\sigma}$	$\ln P(g \hat{\alpha}, \hat{\sigma})$
Belief Propagation	260	0.000574	33.998	-5.15241
Exact	236	0.000652	34.975	-5.17528

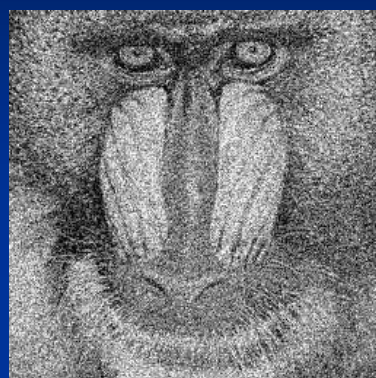
$$\text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2$$

Image Restoration by Gaussian Graphical Model

Original Image



Degraded Image



Belief Propagation



Exact

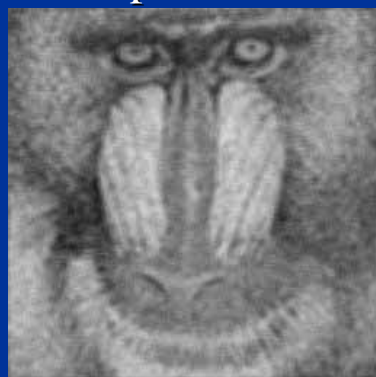


MSE: 1512

MSE: 325

MSE:315

Lowpass Filter



MSE: 411

Wiener Filter



MSE: 545

Median Filter



MSE: 447

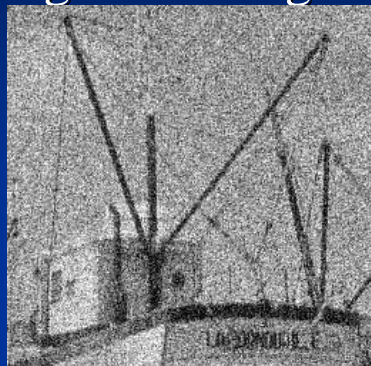
$$1 \text{ MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2$$

Image Restoration by Gaussian Graphical Model

Original Image



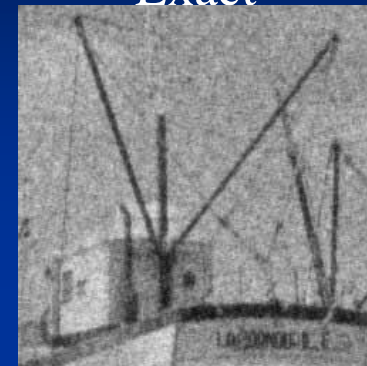
Degraded Image



Belief Propagation



Exact

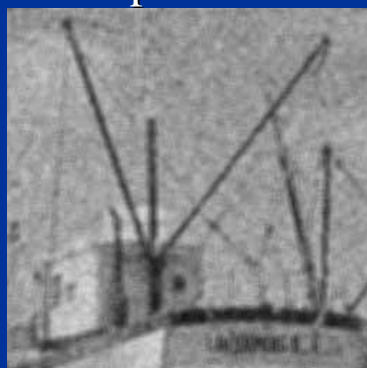


MSE: 1529

MSE: 260

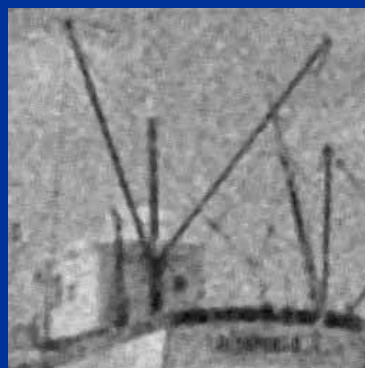
MSE: 236

Lowpass Filter



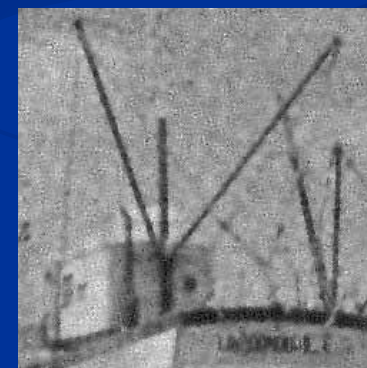
MSE: 224

Wiener Filter



MSE: 372

Median Filter



MSE: 244

$$1 \text{ MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2$$

Extension of Belief Propagation

- **Generalized Belief Propagation**

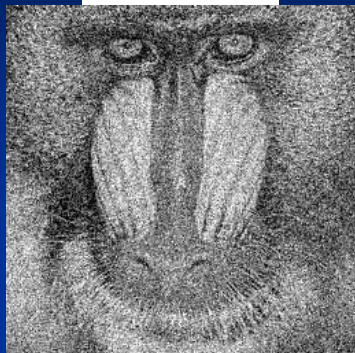
J. S. Yedidia, W. T. Freeman and Y. Weiss: Constructing free-energy approximations and generalized belief propagation algorithms, *IEEE Transactions on Information Theory*, 51 (2005).

- **Generalized belief propagation is equivalent to the cluster variation method in statistical mechanics**

R. Kikuchi: A theory of cooperative phenomena, *Phys. Rev.*, 81 (1951).

T. Morita: Cluster variation method of cooperative phenomena and its generalization I, *J. Phys. Soc. Jpn*, 12 (1957).

$\sigma = 40$



$\sigma = 40$



$$(\hat{\alpha}, \hat{\sigma}) = \arg \max_{(\alpha, \sigma)} P(\bar{g} | \alpha, \sigma)$$

$$\text{MSE} = \frac{1}{|\Omega|} \sum_{i \in \Omega} (f_i - \hat{f}_i)^2$$

	MSE	$\hat{\alpha}$	$\hat{\sigma}$	$\ln P(\mathbf{g} \hat{\alpha}, \hat{\sigma})$
Belief Propagation	327	0.000611	36.302	-5.19201
Generalized Belief Propagation	315	0.000758	37.909	-5.21172
Exact	315	0.000759	37.919	-5.21444

	MSE	$\hat{\alpha}$	$\hat{\sigma}$	$\ln P(\mathbf{g} \hat{\alpha}, \hat{\sigma})$
Belief Propagation	260	0.000574	33.998	-5.15241
Generalized Belief Propagation	236	0.000652	34.971	-5.17256
Exact	236	0.000652	34.975	-5.17528

Image Restoration by Gaussian Graphical Model and Conventional Filters

$\sigma = 40$



GBP



	MSE		MSE
Belief Propagation	327	Lowpass Filter (3x3)	388
		(5x5)	413
Generalized Belief Propagation	315	Median Filter (3x3)	486
		(5x5)	445
Exact	315	Wiener Filter (3x3)	864
		(5x5)	548



(3x3) Lowpass



(5x5) Median



(5x5) Wiener

$$\text{MSE} = \frac{1}{|\Omega|} \sum_{(x,y) \in \Omega} (f_{x,y} - \hat{f}_{x,y})^2$$

Image Restoration by Gaussian Graphical Model and Conventional Filters

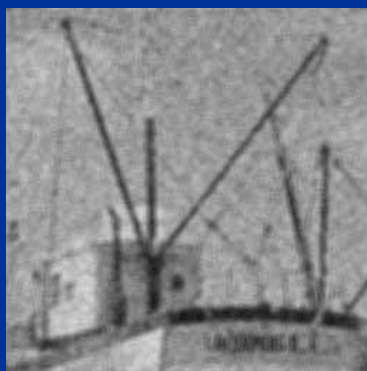
$\sigma = 40$



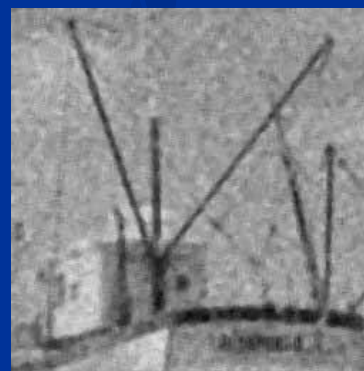
GBP



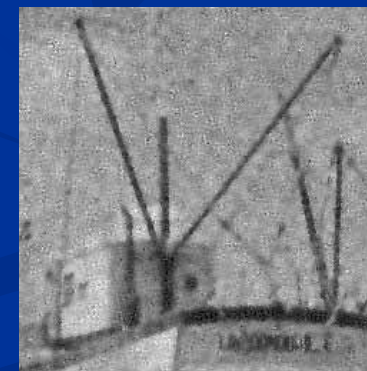
	MSE			MSE
Belief Propagation	260	Lowpass Filter	(3x3)	241
			(5x5)	224
Generalized Belief Propagation	236	Median Filter	(3x3)	331
			(5x5)	244
Exact	236	Wiener Filter	(3x3)	703
			(5x5)	372



(5x5) Lowpass



(5x5) Median



(5x5) Wiener

$$MSE = \frac{1}{|\Omega|} \sum_{(x,y) \in \Omega} (f_{x,y} - \hat{f}_{x,y})^2$$

Contents

1. Introduction
2. Belief Propagation
3. Bayesian Image Analysis and Gaussian Graphical Model
4. Concluding Remarks

Summary

- **Formulation of belief propagation**
- **Accuracy of belief propagation in Bayesian image analysis by means of Gaussian graphical model (Comparison between the belief propagation and exact calculation)**